THE PROJECTIVE TENSOR PRODUCT

OF

COMMUTATIVE BANACH ALGEBRAS

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PREFACE

The material in this thesis is claimed as original, with the exception of those sections where specific mention has been made to the contrary.

The thesis has been composed by myself.

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ABSTRACT

The projective tensor product of two Banach algebras is, in a natural way, a Banach algebra. Two themes are contained in this work.

First, the Arens regularity of the projective tensor product is examined in terms of the Arens regularity of each factor. If this product is Arens regular, then the two factors are Arens regular, except in trivial cases; the converse is not true in general. Certain sufficient conditions for the Arens regularity of the projective tensor product are discussed, and this is applied to completely continuous algebras. The case of some function algebras is also treated.

Second, let V be the projective tensor product of the algebra of complex-valued continuous functions on a compact Hausdorff space with itself, and let $B(H)$ be the algebra of bounded linear operators on a Hilbert space. A natural bicontinuous representation of V on the algebra of bounded linear operators on $B(H)$ is obtained. In addition, there exist similar representations of V on a reflexive Banach space and on the spaces of compact and trace-class operators on a Hilbert space. Finally, the hermitian elements of V are completely determined.

CONTENTS

INTRODUCTION

In the words of Takesaki, tensor products of infinite-dimensional Banach spaces "behave mysteriously" [58:p.229]. The projective tensor product, in particular, has the pleasant property of being a Banach algebra whenever the two factors are Banach algebras; but this algebra is usually rather more complicated than the two algebras one had at the beginning. For example, if B is a closed subalgebra of a Banach algebra A, then the projective tensor product $B \hat{\otimes} A$ of B and A is not necessarily a closed subalgebra of $A \hat{\otimes} A$; if A is a C*-algebra, then $A \hat{\otimes} A$ is no longer a C*-algebra in general; the semisimplicity of A does not ensure that of $A \hat{\otimes} A$; and so on.

In this thesis, the projective tensor product of (usually) commutative Banach algebras is considered. Let A and B be Banach algebras, let Ω be a compact Hausdorff space, and let $C(\Omega)$ be the Banach algebra of continuous complex-valued functions on Ω with the supremum norm. Let H be a Hilbert space, and let $B(H)$ be the Banach algebra of bounded linear operators on H. The main topics discussed here are the regularity of multiplication in the second dual of $A \hat{\otimes} B$ - in other words, the Arens regularity of $A \hat{\otimes} B$; and the existence and properties of a natural bicontinuous representation of $C(\Omega) \hat{\otimes} C(\Omega)$ on $B(H)$. These topics are to some extent inter-related.

Some of the tools originated with Littlewood, who published a paper on bilinear forms in 1930, in which he proved several inequalities [39]. In 1956, Grothendieck produced the "fundamental theorem of the metric theory of tensor products" [24:p.59], which also generalized a result of Littlewood's. The connection between bilinear forms and the projective tensor product in particular is well known, and rests on the observation that the dual $(A\hat{\otimes}B)^*$ of $A\hat{\otimes}B$ is naturally norm-isomorphic to the Banach space of continuous bilinear forms on the Cartesian product of A and B.

One form of Grothendieck's theorem, here Theorem 2.1, comes into play in the construction of the Hilbert space H for the imbedding of $C(\Omega)\hat{\otimes}C(\Omega)$ in $B(B(H))$ mentioned above. In particular, Theorem 2.1 assigns to each element of $C(\Omega)\hat{\otimes}C(\Omega)$ a positive measure μ on Ω, and the Hilbert space H is then the sum of the Hilbert spaces of the form $L^2(\Omega,\mu)$ over all elements of $C(\Omega)\hat{\otimes}C(\Omega)$. Moreover, the imbedding is shown to be bicontinuous, with the norm of the inverse less than or equal to twice the square of the so-called Grothendieck's constant K_G, which lies somewhere between 1 and 2.

In 1951, Arens defined two natural products on the second dual of a Banach algebra [3]. In his paper, he used a result from Littlewood's 1930 work to show that the two products coincide in the second dual of c_0, which is the Banach algebra of complex sequences converging to 0. Unfortunately, as pointed out by Hennefeld in 1968 [25:p.119], Arens's proof contained an error, although his conclusion was correct. It is shown here that the projective tensor product of c_0 with itself behaves in the same way with respect to the Arens products as c_0 does.

Chapter I contains mainly background material. In Section 1, notation and known results are given, together with some proofs. Grothendieck's theorem is proved in Section 2, in the form required for later use. Moreover, a generalization of this theorem due to Pisier is stated as Theorem 2.11. Section 3 is mostly concerned with various equivalent versions of Arens regularity.

Chapter II starts with Section 4, in which a necessary condition for the Arens regularity of $A \hat{\otimes} B$ is given: except in trivial cases, the algebras A and B must be Arens regular if $A \hat{\otimes} B$ is (Theorem 4.1). This is done by looking at the dual of the projective tensor product. The condition is not sufficient, however, and Theorem 4.6, due to Davie, provides a counter-example: $C[0,1] \hat{\otimes} C[0,1]$ is not Arens regular, even though $C[0,1]$ is a C*-algebra, and therefore Arens regular (Theorem 3.7).

It seems that the class of Banach algebras A for which $A \hat{\otimes} A$ is Arens regular remains fairly restricted. In Section 5, completely continuous (CC) algebras are defined; the left and the right multiplication by a fixed element of such an algebra are both compact linear operators. Proposition 5.8 shows that if A and B are CC algebras, then so is $A \hat{\otimes} B$. Next, Theorem 5.13 gives some conditions ensuring the Arens regularity of $A \hat{\otimes} B$ when A and B are CC algebras; one of these involves the linear space structure of A and B only, namely that every bounded linear map from A into B^* should be compact. Theorem 5.13 is then applied to C^*-algebras; if A and B are completely continuous C^*-algebras with A separable, then $A\hat{\otimes}B$ is Arens regular by Theorem 5.19. (In fact, conclusions somewhat stronger follow in both theorems.) In particular, Theorem 5.19 applies to the case $A = B = c_0$. As a by-product of these methods, it is shown in Corollary 5.24 that $A\hat{\otimes}B$ has a quasi-central bounded approximate identity if A and B are C^*-algebras with A being CC and separable; note that in this case it is not known whether $A\hat{\otimes}B$ is necessarily Arens regular.

The projective tensor product of algebras of the form $C(\Omega)$ has important connections with harmonic analysis, as the work of Varopoulos demonstrates [61]. He shows that if G is a compact abelian group, then $C(G)\hat{\otimes}C(G)$ essentially contains $L^1(\hat{G})$. Young proved [62] that $L^1(G)$ is not Arens regular when G is infinite; this is used in Section 6 to prove that $C(\Omega)\hat{\otimes}C(\Omega)$ is not Arens regular if Ω is a compact metric space containing a perfect set.

Carrying this further, it seems that the behaviour of the algebra $C(\Omega)\hat{\otimes}C(\Omega)$ is in some ways similar to that of $L^1(G)$. In his thesis [21 :pp.85,90], Ghahramani showed that the algebra $L^1(G)$ is not isometrically isomorphic to an algebra of operators on a Hilbert space, but that there exists an isometric isomorphism of $L^1(G)$ onto a subalgebra of $B(B(H))$, if G is a locally compact group and $H = L^2(G)$. Størmer [57] discusses certain commutative subalgebras of $B(B(H))$, and also obtains, for G a locally compact abelian group, an isometric isomorphism of $L^1(G)$ onto a commutative subalgebra of $B(B(L^2(G)))$. Here, in Section 7, it is shown that there can be no bicontinuous representation of $V(\Omega) = C(\Omega)\hat{\otimes}C(\Omega)$ on a Hilbert space, at least if Ω is compact, metric, and contains a perfect set (Remark 7.6); but there exists a natural bicontinuous isomorphism of $V(\Omega)$ onto a commutative subalgebra of $B(B(H))$, if Ω is a compact Hausdorff space, and H is a Hilbert space constructed using Grothendieck's theorem. This isomorphism is defined by $\theta(x\otimes y)T = \phi(x)\,T\,\phi(y)$, for x,y in $C(\Omega)$, T in $B(H)$, and ϕ a $*$-representation of $C(\Omega)$ on the Hilbert space H (Theorem 7.4). Moreover, if $C(\Omega)$ is separable, then H can be chosen to be separable (Corollary 7.5).

In Proposition 7.8, a bicontinuous representation of $V(\Omega)$ on a reflexive Banach space is obtained, by using Theorem 7.4. This provides an example of a reflexive space X for which $B(X)$ is not Arens regular; the first example of this kind is due to Young [63]. Young's space is the ℓ^2-sum of $L^{1+\frac{1}{n}}(G)$ for G an infinite compact group, while X here is the ℓ^2-sum of the algebras of the form $B(H_n)$, each H_n being a Hilbert space.

In Section 8, it is proved that there exist bicontinuous representations of $V(\Omega)$ on the spaces of compact and trace-class operators on H, of the same form and norm as that on $B(H)$. The question of the existence of similar results for other Schatten - von Neumann classes of operators $C_p(H)$, where p

satisfies $1 < p < \infty$, $p \neq 2$, remains open. It is only shown that if there is such a result for some C_p , then there is also one for C_q , where $\frac{1}{p} + \frac{1}{q} = 1$ (Proposition 8.6) .

Finally, in Section 9 , another application of Theorem 7.4 yields a complete characterization of the hermitian elements of $C(\Omega) \hat{\otimes} C(\Omega)$; using this, it is possible to determine the hermitian elements of the projective tensor product of any two unital Banach algebras (Note 9.8) . This is due to A.M.Sinclair.

CHAPTER I : BACKGROUND

1. Notation and standard results

In this section, notation, definitions and results used throughout the thesis are established. The knowledge of general functional analysis assumed corresponds to the first 6 chapters of Dunford and Schwartz [19], and that of the Banach algebra theory to the first 17 sections of Bonsall and Duncan [9]. Parts of Dixmier [16], Köthe [34,35] and Rudin [49,50] are also quoted at times. General references for the theory of tensor products are Schatten [51] and Bonsall and Duncan [9 :Ch.6].

1.1 NOTE

Unless otherwise indicated, all linear spaces are assumed complex.

1.2 DEFINITION

An algebra is a linear space A together with the map $A \times A \longrightarrow A$: $(a,b) \longmapsto ab$, called multiplication, satisfying:

$$(ab)c = a(bc)\ ;\quad a(b+c) = ab + ac\ ;\quad (a+b)c = ac + bc\ ;$$

$$(\alpha a)b = \alpha(ab) = a(\alpha b)\ ,\quad \text{for } a,b,c \in A \text{ and } \alpha \in \mathbb{C}\ .$$

An algebra A is commutative if $ab = ba$ for a,b in A.
A normed algebra $(A,\|\cdot\|) = A$ is an algebra A with a norm $\|\cdot\|$ such that $\|ab\| \leq \|a\|\cdot\|b\|$ for $a,b \in A$; a normed algebra is called a Banach algebra if it is complete with respect to the metric induced by the norm. Thus it will be assumed that "the normed algebra A" stands for "the normed algebra $(A,\|\cdot\|)$,

where $\|\cdot\|$ is a specified norm" ; and similarly for normed spaces in general. A normed algebra A is <u>unital</u> if there exists an element $1_A = 1$ of A such that $1a = a1 = a$ $(a \in A)$, and $\| 1 \| = 1$.

<u>1.3 EXAMPLE</u>

The normed space of bounded linear maps from a normed space $(X, \|\cdot\|_X)$ into another $(Y, \|\cdot\|_Y)$ is denoted by $B(X,Y)$; the linear space operations are defined pointwise, and the norm is the operator norm: $\| T \| = \sup \{ \| Tx \|_Y : x \in X , \| x \|_X \leq 1 \}$ for T in $B(X,Y)$. The dual X^* of X is $B(X,\mathbb{C})$. If Y is complete, then $B(X,Y)$ is also complete. Write $B(X)$ for $B(X,X)$; when the multiplication in $B(X)$ is defined by composition, $B(X)$ becomes a unital, non-commutative (in general), normed algebra, and a Banach algebra if X is a Banach space.

<u>1.4 DEFINITION</u>

Let A be a Banach algebra; A is said to be an <u>involutive Banach algebra</u> if there exists a map $* : A \longrightarrow A$, called <u>involution</u> , satisfying:

$$(a+b)^* = a^* + b^* \ ; \quad (a^*)^* = a \ ; \quad (ab)^* = b^*a^* \ ;$$

$$\| a^* \| = \| a \| \ ; \quad (\alpha a)^* = \bar{\alpha} a^* \ , \qquad \text{for } a,b \in A \ , \quad \alpha \in \mathbb{C} \ .$$

An element a of such an algebra is <u>self-adjoint</u> if $a^* = a$. An involutive Banach algebra A is a <u>C*-algebra</u> if , for $a \in A$, $\| a^*a \| = \| a \|^2$.

<u>1.5 EXAMPLE</u>

If $(H, <\cdot,\cdot>)$ is a Hilbert space, then $B(H)$ is a C^*-algebra, with the involution defined by the Hilbert space adjoint: if $T \in B(H)$, then $T^* \in B(H)$ satisfies $< Tx , y > = < x , T^*y >$ for $x,y \in H$. Every C^*-algebra can be identified with a closed $*$-subalgebra of $B(H)$ for some Hilbert space H.

1.6 EXAMPLE

Let Ω be a compact Hausdorff space. The C^*-algebra of continuous maps from Ω into $\mathbb{C}$, with pointwise (therefore commutative) multiplication, and the involution given by $x^*(t) = \overline{x(t)}$ $(x \in C(\Omega),$ $t \in \Omega)$, is denoted by $C(\Omega)$. The norm associated with $C(\Omega)$ is $\| x \| = \sup \{ \, |x(t)| : t \in \Omega \, \}$. Every commutative unital C^*-algebra can be identified with $C(\Omega)$ for some compact Hausdorff Ω.

1.7 DEFINITION

Let A and B be Banach algebras. An (algebra) <u>homomorphism</u> from A into B is a map $\theta \in B(A,B)$ such that $\theta(xy) = \theta(x)\cdot\theta(y)$ for $x,y \in A$. A homomorphism is a <u>monomorphism</u> if it is injective, and an <u>isomorphism</u> if it is injective and surjective. If A and B are unital, and $\theta : A \longrightarrow B$ a homomorphism such that $\theta(1) = 1$, then θ is said to be <u>unital</u>.

A linear map $T \in B(X,Y)$ between Banach spaces X and Y is <u>bicontinuous</u> if there is a constant $c > 0$ such that, for $x \in X$, $\| Tx \| \geq c \| x \|$, and <u>isometric</u> if $\| Tx \| = \| x \|$; X and Y are <u>isometrically isomorphic</u> (as Banach spaces) if there exists an isometric bijection in $B(X,Y)$, and this is written as $X \simeq Y$.

Two Banach algebras, A and B, are <u>isometrically isomorphic</u> (as

Banach algebras) if there exists an isometric (algebra) isomorphism from A to B ; this is denoted by $A \cong B$, and clearly $A \cong B$ implies that $A \simeq B$.

1.8 DEFINITION

A representation of a Banach algebra A on a normed space X is an (algebra) homomorphism from A into $B(X)$.

1.9 EXAMPLE

Let A be a Banach algebra, let $a \in A$, and define $L_a : A \longrightarrow A$ by $L_a(x) = ax$ for $x \in A$. Then the map $a \longmapsto L_a : A \longrightarrow B(A)$ is linear, bounded (of norm $\leqslant 1$), and an (algebra) homomorphism; thus it is a representation of A on A , called the left regular representation. The map $a \longmapsto R_a : A \longrightarrow B(A)$, where $R_a(x) = xa$, is also linear and bounded, but not a homomorphism in general.

1.10 DEFINITION

Let G be a locally compact (Hausdorff) topological group. There exists a non-zero non-negative, regular, left translation invariant measure μ on G , called the Haar measure. The Banach space $L^1(G,\mu) = L^1(G)$ is a Banach algebra, called the group algebra of G , when the product is defined by convolution: for $s \in G$, $f,g \in L^1(G)$,

$$f*g(s) = \int_G f(t)\, g(t^{-1}s)\, d\mu(t) .$$

1.11 DEFINITION

Let X, Y and Z be Banach spaces. A map $F : X \times Y \longrightarrow Z$ is said to be bilinear if it is linear in each coordinate; such a map is

called <u>bounded</u> if there exists a constant $c > 0$ such that $\| F(x,y) \| \leq c \| x \| \cdot \| y \|$ for all $x \in X$, $y \in Y$. The space of all bounded bilinear maps from $X \times Y$ into Z is denoted here by $\mathrm{Bil}(X \times Y, Z)$; this is a Banach space when equipped with the norm
$\| F \|_b = \sup \{ \| F(x,y) \| : x \in X, y \in Y, \| x \| \leq 1, \| y \| \leq 1 \}$. [9 :42.1]

Let $x \in X$, $y \in Y$, and define a rank one linear map $x \otimes y : X^* \longrightarrow Y$ by $(x \otimes y)(f) = f(x) y$ for $f \in X^*$. The linear span of the set $\{ x \otimes y : x \in X, y \in Y \}$ in $B(X^*, Y)$, denoted by $X \otimes Y$, is the <u>algebraic tensor product</u> of X and Y. In particular, if H is a Hilbert space, then $H \otimes H$ is the set of all finite rank operators on H. Similarly, if $f \in X^*$ and $g \in Y^*$, then $f \otimes g$ is a rank one operator from X into Y^* given by $f \otimes g(x) = f(x) g$ for x in X. The tensor product enjoys this universal property:

<u>1.12 PROPOSITION</u> ([9 :42.6])

Let X, Y, Z be Banach spaces. For each bilinear map $F : X \times Y \longrightarrow Z$ there exists a unique linear map $T : X \otimes Y \longrightarrow Z$ such that

$$F(x,y) = T(x \otimes y) \quad \text{for all } (x,y) \in X \times Y . \square$$

<u>1.13 REMARK</u>

Each element of $X \otimes Y$ can be written as $\sum_1^n x_j \otimes y_j$ for some $x_j \in X$, $y_j \in Y$; the sets $\{x_j\}$ and $\{y_j\}$ can be chosen to be linearly independent. [9 :42.3]

<u>1.14 DEFINITION</u>

Let X and Y be Banach spaces. A <u>cross-norm</u> on $X \otimes Y$ is a norm α on $X \otimes Y$ which satisfies $\alpha(x \otimes y) = \| x \| \cdot \| y \|$ for $x \otimes y \in X \otimes Y$.

The <u>projective norm</u> γ on $X \otimes Y$ is defined by

$$\gamma(u) = \inf \{ \sum_1^n \|x_j\| \cdot \|y_j\| : x_j \in X , y_j \in Y , u = \sum_1^n x_j \otimes y_j \}$$

for u in $X \otimes Y$. The completion of $(X \otimes Y, \gamma)$ is denoted by $X \hat{\otimes} Y$, and called the <u>projective tensor product</u> of X and Y .

1.15 <u>REMARK</u>

The projective norm γ is the greatest cross-norm on $X \otimes Y$ in the sense that, if α is another cross-norm, then $\alpha(u) \leqslant \gamma(u)$ for u in $X \otimes Y$. It is possible to define many other interesting norms on $X \otimes Y$; in his Résumé [24] , Grothendieck discusses 14 "natural" tensor norms. In this thesis, however, only the projective tensor product is considered, for the following reason:

1.16 <u>PROPOSITION</u> ([9 :42.18])

Let A and B be Banach algebras. If multiplication on $A \otimes B$ is defined by $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$ for $a_1, a_2 \in A$, $b_1, b_2 \in B$, then $(A \otimes B, \gamma)$ is a normed algebra. Hence $A \hat{\otimes} B$ is a Banach algebra. □

Note that this definition of multiplication on $A \hat{\otimes} B$ is an example of the standard way of defining a linear continuous map from $A \hat{\otimes} B$ into another Banach space : the map is defined on an element $a \otimes b$ in $A \hat{\otimes} B$, and its extension to the whole of $A \hat{\otimes} B$ by continuity and linearity is then tacitly assumed.

If α is a cross-norm different from γ , then $(A \otimes B, \alpha)$ is not, in general, a normed algebra, and hence the completion of $A \otimes B$ with respect to α is not a Banach algebra. It is clear from the definition that the Banach algebra $A \hat{\otimes} B$ is commutative if and only if both A and B are commutative; and if A and B are unital, then so is $A\hat{\otimes}B$, with $1_{A\hat{\otimes}B} = 1_A \otimes 1_B$.

1.17 PROPOSITION

Let X and Y be Banach spaces, let Z be a closed subspace of X.

(i) $X\hat{\otimes}Y \simeq Y\hat{\otimes}X$ (by considering the isometry $x\otimes y \longmapsto y\otimes x$);

(ii) $(X/Z)\hat{\otimes}Y$ is bicontinuously isomorphic to a quotient of $X\hat{\otimes}Y$ [35:41.5(8)];

(iii) if Y is k-dimensional, for some k in $\mathbb{N}$, then $X\hat{\otimes}Y \simeq X^k$ [35:45.1(6)]. □

1.18 COROLLARY

If A and B are Banach algebras, and if I is a closed two-sided ideal of A, then $A\hat{\otimes}B \cong B\hat{\otimes}A$, and $(A/I)\hat{\otimes}B$ is bicontinuously isomorphic to a quotient of $A\hat{\otimes}B$. □

In general, it is not the case that $Z\hat{\otimes}Y$ is a closed subspace of $X\hat{\otimes}Y$ if Z is a closed subspace of X [15:VIII 2.4].

1.19 PROPOSITION ([9:42.13])

Let X and Y be Banach spaces. To each F in $(X\hat{\otimes}Y)^*$, there corresponds an element ϕ of $\mathrm{Bil}(X\times Y,\mathbb{C})$, defined by

$$\phi(x,y) = F(x\otimes y) \quad \text{for } x \text{ in } X \text{ and } y \text{ in } Y,$$

and satisfying $\|\phi\| = \|F\|$.

Conversely, to each ϕ in $\mathrm{Bil}(X\times Y,\mathbb{C})$, there corresponds, by Proposition 1.12, an element F of $(X\hat{\otimes}Y)^*$; and $\|F\| = \|\phi\|$.

Thus $(X\hat{\otimes}Y)^* \simeq \mathrm{Bil}(X\times Y,\mathbb{C})$.

Moreover, $Bil(X\times Y,\mathbb{C}) \simeq B(X,Y^*)$, where the isomorphism is given by $\phi \longmapsto S : Bil(X\times Y,\mathbb{C}) \longrightarrow B(X,Y^*)$, $S(x)(y) = \phi(x,y)$ for x in X and y in Y . Similarly, $Bil(X\times Y,\mathbb{C}) \simeq B(Y,X^*)$. Hence $(X\hat{\otimes}Y)^* \simeq Bil(X\times Y,\mathbb{C}) \simeq B(X,Y^*) \simeq B(Y,X^*)$. □

1.20 DEFINITION

Let X be a normed linear space. The weak topology on X is the weakest (smallest) topology on X with respect to which all elements of X^* are continuous. A net (x_m) in X converges in this topology (or converges weakly) to $x \in X$ if and only if $f(x_m) \longrightarrow f(x)$ for all $f \in X^*$; this is written as $x_m \xrightarrow{w} x$, or $w\text{-}\lim_m x_m = x$.

Similarly, for $x \in X$, let $\pi(x) \in X^{**}$ be the image of x under the canonical injection $\pi : X \longrightarrow X^{**}$, $\pi(x)(f) = f(x)$ $(f \in X^*)$. The weak*-topology on X^* is the weakest topology on X^* with respect to which all elements of $\pi(X) \subset X^{**}$ are continuous. A net $(f_m) \subset X^*$ converges in this topology (or: is w*-convergent) to $f \in X^*$ if and only if $\pi(x)(f_m) \longrightarrow \pi(x)(f)$ for all x in X , which is the same as $f_m(x) \longrightarrow f(x)$ for $x \in X$; this is written as $f_m \xrightarrow{w^*} f$, or $w^*\text{-}\lim_m f_m = f$. It is well known that the closed unit ball of X^* is w*-compact [50 : 3.15] .

1.21 DEFINITION

Let X and Y be Banach spaces. A map $T \in B(X,Y)$ is said to be compact if there exists a subset U of X , containing a neighbourhood of 0 , such that the closure $T(U)^-$ of $T(U)$ in Y is compact. Equivalently, T is compact if $T(U)^-$ is compact in Y

whenever U is a bounded subset of X.
Similarly, T in $B(X,Y)$ is <u>weakly compact</u> if there exists an open set U in X such that the weak closure of $T(U)$ in Y is weakly compact. The class of compact maps in $B(X,Y)$ will be denoted by $K(X,Y)$, and that of weakly compact members of $B(X,Y)$ by $W(X,Y)$. It is well known that $T \in W(X,Y)$ if and only if $T^{**}(X^{**}) \subset \pi(Y)$, where $\pi : Y \longrightarrow Y^{**}$ is the canonical injection [19:VI.4.2].

1.22 DEFINITION

Let X, Y, Z, W be Banach spaces, let $S \in B(X,Y)$ and $T \in B(Z,W)$. Define $S \otimes_\gamma T : (X \otimes Z, \gamma) \longrightarrow (Y \otimes W, \gamma)$ by

$$(S \otimes_\gamma T)(x \otimes z) = (Sx) \otimes (Tz) \quad \text{for } x \in X,\ z \in Z .$$

Then $S \otimes_\gamma T$ extends, by linearity and continuity, to a bounded linear map $S \hat{\otimes} T : X \hat{\otimes} Z \longrightarrow Y \hat{\otimes} W$, and $\| S \hat{\otimes} T \| = \| S \| \cdot \| T \|$; if X, Y, Z, W are Banach algebras and S and T homomorphisms, then $S \hat{\otimes} T$ is also a homomorphism.

1.23 DEFINITION

A Banach space X is said to have the <u>approximation property</u> if, for each Banach space Y, every compact linear map from Y to X is the (operator) norm limit of a sequence of finite rank linear maps in $B(Y,X)$. Equivalently, X has the approximation property if and only if for each compact subset E of X and $\varepsilon > 0$, there exists a finite rank operator $F \in B(X)$ such that $\| Fx - x \| < \varepsilon$ for each $x \in E$; if F can be chosen so that $\| F \| \leq 1$, then X is said to have the <u>metric</u> approximation property.

This concept was introduced by Grothendieck, at least in a formal manner [23] . In general, X possesses this property if X^* does, but not conversely [20] ; for reflexive X , the converse is valid. It is known that if X^* has the approximation property, then for each Banach space Y , every map in $K(X,Y)$ can be approximated by a sequence of finite rank maps in $B(X,Y)$ [15:VIII. 3.6] . The approximation property is not preserved by closed subspaces, nor by quotients. Hilbert spaces, $C(\Omega)$ and $L^p(\Omega)$ spaces, for Ω compact Hausdorff and $1 \leqslant p \leqslant \infty$, all have the approximation property [35:§43].

1.24 DEFINITION

Let A be a Banach algebra. A right approximate identity of A is a net $(a_m)_{m \in M}$ of elements of A , M being a directed set, such that $\| x a_m - x \| \longrightarrow 0$ for every x in A . A left approximate identity of A is defined in the same way, except that $\| a_m x - x \| \longrightarrow 0$ for $x \in A$. A two-sided approximate identity of A is a net in A which is both a left and a right approximate identity. A bounded right (left, two-sided) approximate identity (a_m) of A in addition satisfies $\| a_m \| \leqslant c$ $(m \in M)$ for some constant $c > 0$. It is known that every C*-algebra possesses a two-sided approximate identity bounded by 1 [16:1.7.2] .

The following proposition was formally proved in [40] . The converse also holds, as shown by Robbins [47] and Holub [28] , but will not be used subsequently. The subject of approximate identities is discussed extensively in Doran and Wichmann [17] .

1.25 PROPOSITION

Let A and B be Banach algebras with bounded right approximate identities $(a_m)_{m\in M} \subset A$, $(b_j)_{j\in J} \subset B$. Then $(a_m \otimes b_j)_{(m,j)\in M\times J}$ is a bounded right approximate identity of $A \hat{\otimes} B$.

Proof:

Since $\| a_m \otimes b_j \| = \| a_m \| \cdot \| b_j \|$ for $(m,j)\in M\times J$, and (a_m), (b_j) are bounded, there exists a constant $C > 0$ such that

$\| a_m \otimes b_j \| \leqslant C$ for $(m,j)\in M\times J$. Let $u = \sum_1^n x_k \otimes y_k \in (A\otimes B, \gamma)$.

Now

$$u(a_m \otimes b_j) - u = \sum_{k=1}^{n} \{(x_k a_m - x_k)\otimes y_k + x_k \otimes (y_k b_j - y_k) + (x_k a_m - x_k)\otimes (y_k b_j - y_k)\}$$

for all m,j, and hence $\| u(a_m \otimes b_j) - u \| \longrightarrow 0$.

Suppose $v \in A \hat{\otimes} B$, and let $\varepsilon > 0$; there exists a finite tensor $u = \sum_1^n x_k \otimes y_k$ in $A \hat{\otimes} B$ such that $\| v - u \| < \varepsilon$. Hence, for m,j,

$\| v(a_m \otimes b_j) - v \| \leqslant C\varepsilon + \| u(a_m \otimes b_j) - u \| + \varepsilon$, and therefore $(a_m \otimes b_j)$ is a bounded right approximate identity of $A \hat{\otimes} B$. □

1.26 NOTE

The corresponding result for bounded left approximate identities is clearly also valid, and similarly for (1.29) and (1.30) below.

1.27 DEFINITION

Let A be a Banach algebra, and let X be a linear space. Then X is called a left A-module if there exists a bilinear map from $A\times X$ into X, $(a,x) \longmapsto ax$, such that $a(bx) = (ab)x$ $(a,b\in A,\ x\in X)$.

This map is called module multiplication. The definition of a right A-module is similar, the module multiplication $(a,x) \mapsto xa$ ~~in that case~~ satisfying $(xa)b = x(ab)$ $(x \in X,\ a,b \in A)$. ~~being a bilinear map from $X \times A$ into X~~. If X is a left and a right A-module, and if $(ax)b = a(xb)$ for $a,b \in A$, $x \in X$, then X is said to be an A-bimodule. A right A-module X is a right Banach A-module if X is a Banach space and $\|xa\| \leq C\|x\| \cdot \|a\|$ $(x \in X,\ a \in A)$ for some constant $C > 0$; similar definitions exist for left Banach A-modules and Banach A-bimodules.

1.28 EXAMPLE

A Banach algebra A is a Banach A-bimodule, with the module multiplication coinciding with the multiplication on A.

1.29 THEOREM (Cohen's factorization theorem; [17:17.5],[9:11.12])

Let A be a Banach algebra with a bounded right approximate identity, and let X be a right Banach A-module. For each z in the closure $(XA)^{-} \subset X$ of $XA = \{ xa : x \in X,\ a \in A \}$, and $\delta > 0$, there exist $a \in A$ and $y \in X$ such that

(i) $z = ya$; (ii) $\|y - z\| < \delta$.

In particular, $(XA)^{-} = XA$, so that XA is closed in X. $\square$

1.30 COROLLARY (Varopoulos [60])

Let A be a Banach algebra with a bounded right approximate identity, and let X be a right Banach A-module. Suppose that (z_n) is a sequence in $(XA)^{-}$ such that $z_n \longrightarrow 0$. Then, for each $\delta > 0$, there exists $a \in A$ and a sequence $(y_n) \subset X$ such that (i) $z_n = y_n a$ $(n \in \mathbb{N})$; (ii) $\|y_n - z_n\| < \delta$ $(n \in \mathbb{N})$; (iii) $y_n \longrightarrow 0$.

Proof:

By Theorem 1.29, $(XA)^{-} = XA$. Let $W = \{ (w_n) \subset XA : w_n \longrightarrow 0 \}$ with pointwise addition and scalar multiplication; W is a right Banach A-module when the norm is defined by $\|w\| = \sup_n \|w_n\|$, and the module multiplication by $wa = (w_n a)$ for $w = (w_n) \in W$, $a \in A$. By hypothesis, the sequence $z = (z_n)$ belongs to W.

In order to show that z is an element of $(WA)^{-}$, let $\varepsilon > 0$; since $z_n \longrightarrow 0$, there exists $N \in \mathbb{N}$ such that $\|z_n\| < \varepsilon$ if $n \geq N$. As $z_n \in XA$, write $z_n = x_n a_n$ for $x_n \in X$, $a_n \in A$, $n \in \mathbb{N}$. Let (e_i) be a bounded right approximate identity of A. For fixed n, i,

$$\|z_n e_i - z_n\| = \|x_n a_n e_i - x_n a_n\| \leq \|x_n\| \cdot \|a_n e_i - a_n\| \quad ;$$

since (e_i) is a right bounded approximate identity of A, it follows that $\|z_n e_i - z_n\| \longrightarrow 0$ with i, for each n. Thus there exists $i_0 \in I$ such that, for $n=1,\dots,N$, $\|z_n e_i - z_n\| < \varepsilon$ $(i \geq i_0)$. Since (e_i) is a bounded approximate identity, $\|e_i\| \leq C$ for some $C > 0$. Hence, for $i \geq i_0$,

$$\|ze_i - z\| = \sup_n \|z_n e_i - z_n\| \leq \max_{1 \leq n \leq N} \|z_n e_i - z_n\| + \sup_{n \geq N} \|z_n e_i - z_n\|$$
$$\leq \varepsilon + \sup_{n \geq N} \{\|z_n e_i\| + \|z_n\|\} \leq \varepsilon + C\varepsilon + \varepsilon = \varepsilon(C+2) .$$

Therefore, $z \in (WA)^{-}$.

By Theorem 1.29, there exist $a \in A$ and $y = (y_n) \in W$ such that $z = ya$ and $\|y - z\| < \delta$; thus $z_n = y_n a$ $(n \in \mathbb{N})$, and, for each n, $\|y_n - z_n\| \leq \sup_m \|y_m - z_m\| = \|y - z\| < \delta$, giving (i) and (ii). Finally, $y_n \longrightarrow 0$ by the definition of W, giving (iii). □

1.31 DEFINITION

A (Schauder) basis of a Banach space X is a sequence (x_n) in X

such that for every $x \in X$, there is a unique sequence (α_n) in $\mathbb{C}$ with $x = \sum_1^\infty \alpha_n x_n$. A Banach space with a basis is separable, and has the approximation property [35:43.5.2] .

2. Grothendieck's theorem

This fundamental theorem, as well as a number of its corollaries and equivalent versions, was published in 1956, but remained virtually unknown until the paper by Lindenstrauss and Pełczyński appeared in 1968 [38] . The following version is approximately in [24:3.3.Corollary 4 , 4.2.Theorem 3] .

2.1 THEOREM

Let Ω be a compact Hausdorff space. There exists a universal constant K_G, $K_G \leq 2 \sinh \frac{\pi}{2}$ (< 5) , with the following property: for each bounded bilinear map $F : C(\Omega) \times C(\Omega) \longrightarrow \mathbb{C}$, there is a probability measure μ on Ω such that

$$|F(x,y)| \leq 2K_G^2 \| F \| \left(\int_\Omega |x|^2 \, d\mu\right)^{\frac{1}{2}} \left(\int_\Omega |y|^2 \, d\mu\right)^{\frac{1}{2}} \qquad (x,y \in C(\Omega)).$$

Thus, if $C(\Omega)$ is regarded as a dense subspace of $L^2(\Omega,\mu)$, then F extends by continuity to a bilinear map F' in the space $\mathrm{Bil}(L^2(\Omega,\mu) \times L^2(\Omega,\mu) \,,\, \mathbb{C})$, and $\| F' \| \leq 2K_G^2 \| F \|$.

(A probability measure μ on Ω is a positive measure such that $\mu(\Omega) = 1$ [50:3.26].)

2.2 NOTE

The exact value of the best possible universal constant is unknown. Let k_G denote the corresponding constant in the case when the underlying field is $\mathbb{R}$. It is known that k_G and K_G can be chosen to satisfy $1 < K_G < \frac{\pi}{2} \leq k_G < 1.783$ [36;43]. The proof of Theorem 2.1 given here is due mainly to Lindenstrauss & Pełczyński.

2.3 DEFINITION

Let X and Y be Banach spaces, let $T \in B(X,Y)$, and suppose that $1 \leq p < \infty$. The operator T is said to be p-absolutely summing if there exists a constant $C > 0$ such that the following inequality holds for every finite set $\{x_1, \ldots, x_n\}$ in X:

$$\sum_1^n \|Tx_i\|^p \leq C^p \sup \left\{ \sum_1^n |x^*(x_i)|^p : x^* \in X^*, \ \|x^*\| \leq 1 \right\}.$$

Let $\pi_p(T)$ be the infimum of all such constants C over finite subsets of X, and define $\Pi_p(X,Y) = \{ T \in B(X,Y) : \pi_p(T) < \infty \}$. Then $\Pi_p(X,Y)$ is a normed linear space, with the norm $\pi_p(\cdot)$ [42].

2.4 THEOREM (Pietsch [42:Theorem 4.2], also [38:Proposition 3.1])

Let X and Y be Banach spaces, let $T \in B(X,Y)$, and $1 \leq p < \infty$. Then T is p-absolutely summing if and only if there exists a probability measure μ on the set Q = the w*-closure of the set of extreme points of $\{x^* \in X^* : \|x^*\| \leq 1\}$, such that

$$\|Tx\| \leq \pi_p(T) \cdot \left(\int_Q |q(x)|^p \, d\mu(q) \right)^{\frac{1}{p}} \quad \text{for all } x \in X.$$

Proof:

If such a probability measure exists, then T is clearly p-absolutely summing by definition. Conversely, suppose that $\pi_p(T) < \infty$, and define the subset W of $C(Q)$ by

$$W = \{\, g \in C(Q) : \text{there exists } (x_i)_1^n \subset X \text{ such that } \sum_1^n \|Tx_i\|^p = 1$$
$$\text{and } g(q) = (\pi_p(T))^p \cdot \sum_1^n |q(x_i)|^p \quad \text{for } q \in Q \,\} .$$

Then W is convex: indeed, if g and h belong to W, then, for some $(x_i)_1^n$, $(y_j)_1^m$ in X with $\sum_1^n \|Tx_i\|^p = \sum_1^m \|Ty_j\|^p = 1$, and for $0 < \lambda < 1$, $q \in Q$,

$$(\lambda g + (1-\lambda)h)(q) = (\pi_p(T))^p \left(\sum_1^n |\lambda^{\frac{1}{p}} \cdot q(x_i)|^p + \sum_1^m |q((1-\lambda)^{\frac{1}{p}} y_j)|^p \right)$$

where $\sum_1^n \|T(\lambda^{\frac{1}{p}} \cdot x_i)\|^p + \sum_1^m \|T((1-\lambda)^{\frac{1}{p}} y_j\|^p = \lambda + (1-\lambda) = 1$. Thus $\lambda g + (1-\lambda)h$ belongs to W.

Now let $M = \{\, f \in C(Q) : \sup_{q \in Q} \operatorname{Re} f(q) < 1 \,\}$, where $\operatorname{Re} z$ is the real part of the complex number z. If $g \in W \cap M$, then

$$1 > \sup_{q \in Q} \operatorname{Re} g(q) = \sup_{q \in Q} (\pi_p(T))^p \sum_1^n |q(x_i)|^p ,$$

for some sequence $(x_i)_1^n$ in X with $\sum_1^n \|Tx_i\|^p = 1$;

$$\geqslant \sup \{ (\pi_p(T))^p \sum_1^n |x^*(x_i)|^p : x^* \in X^* , \|x^*\| \leqslant 1 \},$$ for by the Krein-Milman theorem, $\mathcal{U} \overset{\text{def}}{=} \{x^* \in X^* : \|x^*\| \leq 1\} = \overline{co}\, Q$; hence, if $x^* \in \mathcal{U}$, then there are $q_1, \dots, q_m \in Q$, $\alpha_1, \dots, \alpha_m \in \mathbb{C}$ s.t. $x^* \approx \sum \alpha_k q_k$ and $\sum |\alpha_k| = 1$. Then $(\sum_i |x^*(x_i)|^p)^{\frac{1}{p}} \approx (\sum_{i=1}^n | \sum_{k=1}^m \alpha_k q_k(x_i)|^p)^{\frac{1}{p}} \leq \sum_{k=1}^m |\alpha_k| (\sum_{i=1}^n |q_k(x_i)|^p)^{\frac{1}{p}}$, by the triangle inequality in ℓ^p_n;

$$\geqslant \sum_1^n \|Tx_i\|^p , \quad \text{by the definition of } \pi_p(T) ;$$

$$= 1 : \text{ contradiction.}$$

Hence the sets W and M are disjoint.

The set M is certainly open and convex. Since W and M are non-empty, disjoint, convex, and M is open in $C(Q)$, by the Hahn-Banach separation theorem [50:3.4] there exists a functional ψ in $B(C(Q),\mathbb{C})$ and a real number ζ such that $\operatorname{Re} \psi(f) < \zeta \leqslant \operatorname{Re} \psi(g)$ for all $f \in M$ and $g \in W$; thus $0 = \operatorname{Re} \psi(0) < \zeta$, since $0 \in M$.

Let $\phi = \operatorname{Re}\psi$, then $0 \neq \phi \in B(C(Q), \mathbb{R})$, and by replacing ϕ by $\frac{1}{\zeta}\phi$, assume that $\phi(f) < 1 \leqslant \phi(g)$ for all $f \in M$, $g \in W$.

Now if $f \in C(Q)$ and $\|f\| < 1$, then $\sup\limits_{q \in Q} \operatorname{Re} f(q) \leqslant \sup\limits_{q \in Q} |f(q)| < 1$, and hence $\{f \in C(Q) : \|f\| < 1\} \subset M$; thus $\phi(1) \leqslant 1$, where $1(q) = 1$ for all $q \in Q$. Also, if $f \in C(Q)$ is positive, that is, $f(q) \in \mathbb{R}$ and $f(q) \geqslant 0$ for all $q \in Q$, then $-f \in M$, and $\alpha(-f) \in M$ for $\alpha > 0$; hence, for $\alpha > 0$,

$$\phi(\alpha(-f)) < 1 \quad \Rightarrow \quad \alpha\phi(f) > -1 \quad \Rightarrow \quad \phi(f) > -\frac{1}{\alpha} \quad ,$$

and so $\phi(f) \geqslant 0$. Thus ϕ is a positive real functional on $C(Q)$. By the Riesz representation theorem [49: 2.14] , there exists a (real) positive measure ν on Q such that

$$\phi(f) = \int_Q f \, d\nu \quad \text{for } f \text{ in } C(Q) \ ; \quad \|\phi\| = \nu(Q) \ .$$

Since $\phi \neq 0$, it follows that

$$0 < \|\phi\| = \nu(Q) = \int_Q 1 \, d\nu = \phi(1) \leqslant 1 \ ;$$

hence, if the measure μ is defined by $\mu = \dfrac{1}{\nu(Q)}\nu$ on Q , then μ is positive, and $\mu(Q) = 1$; thus μ is a probability measure.

Finally, let $x \in X$; if $Tx = 0$, then the inequality required is trivially satisfied ; otherwise, the function $g \in C(Q)$ defined by $g(q) = (\|Tx\|^{-1} \cdot \pi_p(T) \cdot |q(x)|)^p$ for $q \in Q$ belongs to W , and so $1 \leqslant \phi(g) = \int_Q g \, d\nu \leqslant \int_Q g \, d\mu$ yields $\|Tx\|^p \leqslant (\pi_p(T))^p \int_Q |q(x)|^p d\mu(q)$.□

<u>2.5 COROLLARY</u> ([38:3.Corollary 1])

Let Ω be a compact Hausdorff space , and let $X = C(\Omega)$.

If T is a 2-absolutely summing operator from X into X^*, then there exists a probability measure μ on Ω, and a linear map S in $B(L^2(\Omega,\mu), X^*)$ with $\|S\| = \pi_2(T)$, so that T factors: $C(\Omega) \xrightarrow{J} L^2(\Omega,\mu) \xrightarrow{S} C(\Omega)^*$, where J is the natural imbedding.

Proof:

If $X = C(\Omega)$ and Q is defined as in Theorem 2.4, then [19: Lemma V.8.6] shows that Q is canonically isomorphic to Ω. Thus Theorem 2.4 gives the existence of μ, and a bounded linear map S_0 from $(J(X))^-$ into X^* with $\|S_0\| = \pi_2(T)$ and $T = S_0 J$. Since $(J(X))^-$ is a closed subspace of the Hilbert space $L^2(\Omega,\mu)$, there exists a projection P of $L^2(\Omega,\mu)$ onto $(J(X))^-$. Define $S \in B(L^2(\Omega,\mu), X^*)$ by $S = S_0 P$, then $\|S\| = \pi_2(T)$. □

2.6 PROPOSITION

Let Ω be a compact Hausdorff space, and let $X = C(\Omega)$. Then every bounded linear map from X into X^* is 2-absolutely summing, and there exists a constant $K_6 \leq 2 \sinh \frac{\pi}{2}$ such that $\pi_2(T) \leq K_6 \|T\|$ for all $T \in B(X,X^*) = \Pi_2(X,X^*)$.

Proof of Theorem 2.1, assuming Proposition 2.6:

Let $F \in \mathrm{Bil}(C(\Omega) \times C(\Omega), \mathbb{C})$; using the standard identification (Proposition 1.19), regard F as an element of $B(C(\Omega),C(\Omega)^*)$. By Proposition 2.6, F belongs to $\Pi_2(C(\Omega),C(\Omega)^*)$, and there is a (universal) constant K_6 such that $\pi_2(F) \leq K_6\|F\|$. By Corollary 2.5, there exists a probability measure μ on Ω and a map $S \in B(L^2(\Omega,\mu), C(\Omega)^*)$ with $\|S\| = \pi_2(F) \leq K_6\|F\|$, so that $F = SJ$.

Let $H = L^2(\Omega,\mu)$. For each y_0 in $C(\Omega)$, the map $S'(\cdot)(y_0) : H^* \longrightarrow \mathbb{C}$ is bounded and linear, where $S' \in B(H^*, C(\Omega)^*)$ is defined by $S'(x^*)(\cdot) = \overline{S(x)(\cdot)}$ for $x^* \in H^*$, $x^*(\cdot) = \langle \cdot , x \rangle$. Hence $S'(\cdot)(y_0) \in H^{**} \simeq H$ for each $y_0 \in C(\Omega)$, and so $S'(x^*)(y_0) = x^*(Ty_0)$ for some $T \in B(C(\Omega),H)$; thus S' is the adjoint of T , and $\|T\| = \|T^*\| = \|S'\| = \|S\| \leq K_G\|F\|$.

The operator $T \in B(C(\Omega),H)$ factorizes by Proposition 2.6A: there is a probability measure ν on Ω and a map R s.t. $T: C(\Omega) \xrightarrow{J'} L^2(\nu) \xrightarrow{R} L^2(\mu)$, and $\|R\| \leq K_G\|T\| \leq K_G^2\|F\|$. Hence, for x, y in $C(\Omega)$,

$$|F(x)(y)| = |(SJx)(y)| \leq \|Jx\| \cdot \|RJ'y\| \leq \|R\| \cdot \|Jx\| \cdot \|J'y\| \leq$$

$$\leq 2K_G^2 \|F\| \cdot \|x\|_{L^2(\lambda)} \|y\|_{L^2(\lambda)} \text{, as required, if } \lambda = \frac{\mu+\nu}{2} . \square$$

In order to prove Proposition 2.6 , several additional results are needed.

<u>2.7 LEMMA</u> ([38:Lemma 2.1])

Let $\ell^2(n)$ be the <u>real</u> n-dimensional Hilbert space, and let $S^{n-1} = \{x \in \ell^2(n) : \|x\| = 1\}$. Let $|\cdot|$ be the Lebesgue measure, and let m be a multiple of the Lebesgue measure on S^{n-1} adjusted so that $m(S^{n-1}) = 1$. Let $x, y \in S^{n-1}$, and define the number c by $c = \int_{S^{n-1}} \operatorname{sgn} \langle x , u \rangle \operatorname{sgn} \langle y , u \rangle \, dm(u)$.

Then $\langle x , y \rangle = \sin (\frac{\pi}{2}c)$.

<u>Proof</u>:

Since m is rotation-invariant, it is possible to choose a basis in $\ell^2(n)$ so that $x = (1,0,\ldots,0)$ and $y = (\cos\theta, \sin\theta, 0, \ldots, 0)$, where θ is the unique number satisfying $0 \leq \theta \leq \pi$, $\cos\theta = \langle x , y \rangle$.

2.6A PROPOSITION ([38: 4.2. Theorem 3] 5. Cor 1)

Let Ω be a compact Hausdorff space, and let μ be a probability measure on Ω. If $T: C(\Omega) \to L^2(\Omega,\mu)$ is a bounded linear map, then there exists a probability measure ν on Ω and a bounded linear map $R: L^2(\Omega,\nu) \to L^2(\Omega,\mu)$ s.t. $T = RJ'$, where $J': C(\Omega) \to L^2(\Omega,\nu)$ is the natural imbedding, and $\|R\| \leq K_G\|T\|$, where K_G is Grothendieck's constant.

Now change to polar coordinates: write $\phi = (\phi_1,\ldots,\phi_{n-1})$ to express $u \in S^{n-1}$ as $u(\phi) = (u^1(\phi),\ldots,u^n(\phi))$, with

$$u^1(\phi) = \prod_1^{n-1} \sin \phi_i \quad , \quad u^n(\phi) = \cos \phi_{n-1} \quad , \text{ and}$$

$$u^k(\phi) = (\cos \phi_{k-1}) \prod_{i=k}^{n-1} \sin \phi_k \qquad \text{for } k=2,\ldots,n-1 .$$

By the standard method of change of variable [49:8.27], for each bounded measurable function g on S^{n-1},

$$\int_{S^{n-1}} g(u)\, dm(u) = |S^{n-1}|^{-1} \int_{I^{n-1}} g(u(\phi))\cdot J(\phi)\, d(\phi), \quad \ldots\ldots(1)$$

where $I^{n-1} = \{\phi\colon 0 \leq \phi_1 < 2\pi\ ,\ 0 \leq \phi_k \leq \pi \ \ (2 \leq k \leq n-1)\}$, J is the Jacobian: $J(\phi) = \prod_2^{n-1} (\sin \phi_i)^{i-1}$, and $|S^{n-1}| = \int_{I^{n-1}} J(\phi)\, d\phi$.

Now take $g(u) = \operatorname{sgn}(\langle x, u\rangle\langle y, u\rangle) = \operatorname{sgn}(u^1(u^1\cos\theta + u^2\sin\theta))$ to obtain:

$$g(u(\phi)) = \operatorname{sgn}\{(\prod_1^{n-1}\sin\phi_i)[(\prod_1^{n-1}\sin\phi_i)\cos\theta + (\cos\phi_1)(\prod_2^{n-1}\sin\phi_i)\sin\theta]\}$$

$$= \operatorname{sgn}\{(\prod_2^{n-1}\sin\phi_i)^2(\sin\phi_1\cdot\sin\phi_1\cdot\cos\theta + \cos\phi_1\cdot\sin\theta\cdot\sin\phi_1)\}$$

$$= \operatorname{sgn}\{\sin\phi_1\cdot\sin(\phi_1+\theta)\}$$

$$= \begin{cases} 1\ , & \text{if } \ \phi_1 \in (0,\pi-\theta)\cup(\pi,2\pi-\theta)\ ; \\ -1\ , & \text{if } \ \phi_1 \in (\pi-\theta,\pi)\cup(2\pi-\theta,2\pi); \\ 0\ , & \text{otherwise}\ . \end{cases}$$

Therefore, using (1) ,

$$c = \int_{S^{n-1}} g(u)\, dm(u) =$$

$$= |S^{n-1}|^{-1} \left(\int_0^{2\pi} \operatorname{sgn}\{\sin\phi_1\cdot\sin(\phi_1+\theta)\}\, d\phi_1\right)\cdot\prod_2^{n-1}\left(\int_0^{\pi}(\sin\phi_i)^{i-1}\, d\phi_i\right) ;$$

since $|S^{n-1}| = \int_{I^{n-1}} J(\phi)\, d\phi = 2\pi \prod_{2}^{n-1} \int_0^{\pi} (\sin\phi_i)^{i-1}\, d\phi_i$, it follows that

$$c = \frac{1}{2\pi}(2\pi - 4\theta) = 1 - \frac{2\theta}{\pi} .$$

Thus $\langle x , y \rangle = \cos\theta = \cos\left(\frac{\pi}{2}(1-c)\right) = \sin\left(\frac{\pi}{2}c\right)$. □

The following result, Grothendieck's inequality, implies Theorem 2.1 , but is used here only as a lemma for proving the theorem.

2.8 THEOREM ([38:4.Corollary 1])

Let (a_{ij}) , $i,j=1,\ldots,n$, be a real matrix such that there exists a constant $M > 0$ satisfying $\left|\sum_{i,j=1}^{n} a_{ij}s_i t_j\right| \leq M$ whenever s_i , t_j are real numbers and $|s_i|$, $|t_j| \leq 1$ for all i,j . Then, for every $2n$ elements $x_1,\ldots,x_n,y_1,\ldots,y_n$ of an arbitrary real Hilbert space $(H, \langle\cdot,\cdot\rangle)$,

$$\left| \sum_{i,j=1}^{n} a_{ij} \langle x_i , y_j \rangle \right| \leq M \left(\sinh \frac{\pi}{2}\right)\cdot \max_i \|x_i\| \cdot \max_j \|y_j\| .$$

Proof: [12]

Every $2n$ elements of H belong to a $2n$-dimensional linear subspace of H isometric to $\ell^2(2n)$. Hence assume, without loss of generality, that $x_1,\ldots,x_n,y_1,\ldots,y_n$ belong to $\ell^2(2n)$.

Consider first the case when these elements are in S^{2n-1} , and apply Lemma 2.7 to each pair i,j :

$$\langle x_i , y_j \rangle = \sin\left(\frac{\pi}{2} c_{ij}\right) ,$$

where $c_{ij} = \int_{S^{2n-1}} \operatorname{sgn} \langle x_i , u \rangle \operatorname{sgn} \langle y_j , u \rangle \, dm(u)$.

Thus $\left|\sum_{i,j=1}^{n} a_{ij} < x_i , y_j >\right| = \left|\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+1} \cdot \sum_{i,j=1}^{n} a_{ij} c_{ij}^{2k+1}\right|$

$$\leq \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+1} \left| \sum_{i,j=1}^{n} a_{ij} c_{ij}^{2k+1} \right| .$$

Now

$$\left|\sum_{i,j=1}^{n} a_{ij} c_{ij}^{2k+1}\right| = \left|\sum_{i,j=1}^{n} a_{ij} \prod_{\ell=1}^{2k+1} \int_{S^{2n-1}} \operatorname{sgn}< x_i , u_\ell > \operatorname{sgn}< y_j , u_\ell > dm(u_\ell)\right|$$

$$\leq \int_{(S^{2n-1})^{2k+1}} \left|\sum_{i,j=1}^{n} a_{ij}\left(\prod_{\ell=1}^{k} \operatorname{sgn}< x_i , u_\ell >\right)\left(\prod_{\ell=1}^{k} \operatorname{sgn}< y_j , u_\ell >\right)\right| dm^{2k+1}(u_1,\ldots,u_{2k+1})$$

$\leq M$ by hypothesis, since $\left|\prod_{\ell=1}^{k} \operatorname{sgn}< x_i , u_\ell >\right| , \left|\prod_{\ell=1}^{k} \operatorname{sgn}< y_j , u_\ell >\right| \leq 1.$

Therefore, $\left|\sum_{i,j=1}^{n} a_{ij} < x_i , y_j>\right| \leq M \sinh \frac{\pi}{2}$, as required .

To show that the result holds for arbitrary x_i , y_j in $\ell^2(2n)$, and not only for elements of S^{2n-1}, observe that for arbitrary real numbers $b_1,\ldots,b_n,d_1,\ldots,d_n$, the matrix $(a_{ij}b_i d_j)$ is such that $\left|\sum_{i,j=1}^{n} a_{ij}b_i d_j s_i t_j\right| \leq M_1$ whenever $s_i, t_j \in \mathbb{R}$ and $|s_i| , |t_j| \leq 1$, where $M_1 = M \cdot \max_i |b_i| \cdot \max_j |d_j|$: indeed, the case $\max_i |b_i| \cdot \max_j |d_j| = 0$ is trivial ; otherwise, by the hypothesis of the theorem,

$$\left|\sum_{i,j=1}^{n} a_{ij} \frac{b_i s_i}{\max_k |b_k|} \frac{d_j t_j}{\max_\ell |d_\ell|}\right| \leq M , \quad \text{for } s_i , t_j \in \mathbb{R} , |s_i| , |t_j| \leq 1.$$

Now let $x_1,\ldots,x_n,y_1,\ldots,y_n$ belong to $\ell^2(2n)$; setting $b_i = \|x_i\|$, $d_j = \|y_j\|$ in the preceding paragraph, it follows that

$$\left|\sum_{i,j=1}^{n} (a_{ij}\|x_i\| \, \|y_j\|) s_i t_j\right| \leq M_1 \qquad (s_i, t_j \in \mathbb{R} , |s_i| , |t_j| \leq 1).$$

Taking $\frac{x_i}{\|x_i\|}$, $\frac{y_j}{\|y_j\|}$ in S^{2n-1} if $\|x_i\|$, $\|y_j\| \neq 0$, otherwise an arbitrary element of S^{2n-1} , apply the proved part of the theorem to obtain:

$$\left|\sum_{i,j=1}^{n} a_{ij} < x_i , y_j > \right| = \left|\sum_{i,j=1}^{n} (a_{ij}\|x_i\| \ \|y_j\| \ ) < \frac{x_i}{\|x_i\|} , \frac{y_j}{\|y_j\|} > \right|$$

$$\leq M_1 \ \sinh \frac{\pi}{2}$$

$$= M \ (\sinh \frac{\pi}{2})\cdot \max_i \|x_i\| \cdot \max_j \|y_j\| \quad . \square$$

2.9 COROLLARY

Let (a_{ij}) , $i,j=1,\ldots,n$, be a complex matrix such that there exists a constant $M > 0$ satisfying $\left|\sum_{i,j=1}^{n} a_{ij}s_i t_j\right| \leq M$ whenever s_i , t_j are complex numbers and $|s_i|$, $|t_j| \leq 1$. Then, for every $2n$ elements of an arbitrary Hilbert space $(H, <\cdot,\cdot>)$,

$$\left|\sum_{i,j=1}^{n} a_{ij} < x_i , y_j >\right| \leq 2M \ (\sinh \frac{\pi}{2})\cdot \max \|x_i\| \cdot \max \|y_j\| \quad .$$

Proof:

Let $(b_{ij}) = (\mathrm{Re}\ a_{ij})$, $(d_{ij}) = (\mathrm{Im}\ a_{ij})$; then, whenever s_i, t_j are <u>real</u> and $|s_i|$, $|t_j| \leq 1$,

$$\left|\sum_{i,j=1}^{n} b_{ij}s_i t_j\right| \leq M \quad \text{and} \quad \left|\sum_{i,j=1}^{n} d_{ij}s_i t_j\right| \leq M \ .$$

Thus the hypotheses of Theorem 2.8 are satisfied by (b_{ij}) and (d_{ij}) . The conclusion required follows if and only if the following statement is true: for every $(x_i)_1^n$ in a Hilbert space,

$$\sum_{j=1}^{n} \left\| \sum_{i=1}^{n} a_{ij} \ x_i \right\| \leq 2M \ (\sinh \frac{\pi}{2}) \ \max_i \|x_i\| \quad . \qquad \ldots\ldots(1)$$

Indeed, if the conclusion is valid, and $x_1,\dots,x_n$ belong to an arbitrary Hilbert space H, then, using the Hahn-Banach theorem, choose $y_1,\dots,y_n$ in H such that $\|y_j\| = 1$ and

$$< \sum_{i=1}^n a_{ij}x_i \, , \, y_j > \; = \|\sum_{i=1}^n a_{ij}x_i\| \qquad \text{for } j=1,\dots,n \, .$$

Then $\sum_{j=1}^n \|\sum_{i=1}^n a_{ij}\, x_i\| = |\sum_{i,j=1}^n a_{ij} < x_i \, , y_j > | \leq 2M\,(\sinh \frac{\pi}{2})\cdot \max_i \|x_i\|$, which is (1). Conversely, if (1) holds and $(x_i), (y_j) \in H$, then

$$|\sum_{i,j=1}^n a_{ij} < x_i \, , y_j > | \leq \sum_{j=1}^n \|\sum_{i=1}^n a_{ij}\, x_i\| \, \|y_j\| \leq 2M(\sinh \frac{\pi}{2})\cdot\max_i \|x_i\|\cdot\max_j \|y_j\|$$

which is the conclusion of the corollary.

Now suppose $x_1,\dots,x_n$ belong to a Hilbert space; applying the corresponding version of (1) in the real case,

$$\sum_{j=1}^n \|\sum_{i=1}^n a_{ij}\, x_i\| \leq \sum_{j=1}^n (\|\sum_{i=1}^n b_{ij}x_i\| + \|\sum_{i=1}^n d_{ij}\, x_i\|)$$

$$\leq 2M\,(\sinh \frac{\pi}{2})\cdot \max_i \|x_i\| \quad , \text{ which is (1).} \square$$

2.10 NOTE

If, under the hypotheses of Theorem 2.8, the real constant k_G is defined by

$$k_G = \inf \{c>0 : |\sum_{i,j=1}^n a_{ij} < x_i \, , y_j > | \leq cM\cdot\max_i \|x_i\|\cdot\max_j \|y_j\| \, , \; x_i \text{ and } y_j$$
$$\text{for } i,j=1,\dots,n \text{ belong to a real Hilbert space }, \; n \in \mathbb{N} \} \, ,$$

then Theorem 2.8 shows that $k_G \leq \sinh \frac{\pi}{2}$. If, in the complex case, the constant K_G is similarly defined, then Corollary 2.9 shows that $K_G \leq 2\sinh\frac{\pi}{2}$.

<u>Proof of Proposition 2.6</u>: [38:Theorem 4.3]

Let $(x_i)_1^n$ in $X = C(\Omega)$ be a finite set, and by scaling if necessary, assume $\sum_1^n \|x_i\|^2 \leq 1$; then $\sum_1^n |x^*(x_i)|^2 \leq \|x^*\|^2$ $(x^* \in X^*)$.

Without loss of generality, also assume that the x_i's are linearly independent; then there is a constant $M > 0$ such that

$$\sum_1^n |a_i| \leq M \left\| \sum_1^n a_i x_i \right\| \qquad (a_i \in \mathbb{C}) .$$

Let $\varepsilon > 0$. By using a partition of unity, there exists an m-dimensional subspace W of $C(\Omega)$, isometric to $\ell^\infty(m)$, and such that $\|x_i - w_i\| < \varepsilon$ for some linearly independent $w_1, \ldots, w_n$ in W if ε is small enough. Let $P : X \longrightarrow \operatorname{span}\{w_i\}$ be a projection with $\|P\| \leq n$. Now choose $y_{n+1}, \ldots, y_m$ in X so that each $y_j \in \operatorname{Ker} P$, and $\{w_1, \ldots, w_n, y_{n+1}, \ldots, y_m\}$ is a basis for W .

Set $W_0 = \operatorname{span}\{x_1, \ldots, x_n, y_{n+1}, \ldots, y_m\}$, then $\operatorname{span}\{x_i\} \subset W_0$. Define a map S from $W \simeq \ell^\infty(m)$ onto W_0 by

$$S\left(\sum_1^n a_i w_i + \sum_{n+1}^m b_j y_j \right) = \sum_1^n a_i x_i + \sum_{n+1}^m b_j y_j \qquad (a_i, b_j \in \mathbb{C}) .$$

Let $w = \sum_1^n a_i w_i + \sum_{n+1}^m b_j y_j$ belong to W . From the inequality

$$\left\| \sum_1^n a_i x_i - \sum_1^n a_i w_i \right\| \leq \varepsilon \sum_1^n |a_i| \leq \varepsilon M \left\| \sum_1^n a_i x_i \right\| ,$$

and using the definition of P , it follows that

$$\begin{aligned} \left\| \sum_1^n a_i x_i \right\| &\leq \left\| \sum_1^n a_i x_i - \sum_1^n a_i w_i \right\| + \left\| \sum_1^n a_i w_i \right\| \\ &\leq \varepsilon M \left\| \sum_1^n a_i x_i \right\| + \|P(w)\| \\ &\leq \varepsilon M \left\| \sum_1^n a_i x_i \right\| + n \|w\| . \end{aligned}$$

Thus, if $\varepsilon M < 1$, then $(1 - \varepsilon M) \left\| \sum_1^n a_i x_i \right\| \leq n \|w\|$.

Hence $\| \sum_1^n a_i x_i - \sum_1^n a_i w_i \| \leq \frac{\varepsilon n M}{1-\varepsilon M} \| w \|$. Therefore,

$$\| Sw \| = \| \sum_1^n a_i x_i + \sum_{n+1}^m b_j y_j \| \leq$$

$$\leq \| \sum_1^n a_i x_i - \sum_1^n a_i w_i \| + \| \sum_1^n a_i w_i + \sum_{n+1}^m b_j y_j \|$$

$$\leq \frac{\varepsilon n M}{1-\varepsilon M} \| w \| + \| w \|$$

$$= (1 + \frac{\varepsilon n M}{1-\varepsilon M}) \| w \| .$$

Thus $\| S \| \leq 1 + \frac{nM\varepsilon}{1-\varepsilon M}$, and similarly $\|S^{-1}\| \leq 1 - \frac{\varepsilon n M}{1-\varepsilon M}$. Hence there exists an invertible map S in $B(\ell^\infty(m), W_0)$ such that $\text{span}\{x_i\} \subset S(\ell^\infty(m)) = W_0$, and $\| S \| = 1$, $\|S^{-1}\| \leq 1 + \varepsilon$.

If T is the given bounded linear map from X into X^* , then $T(S(\ell^\infty(m)))$ is finite-dimensional in X^* . Now $X^* = C(\Omega)^*$ is isometric to $L^1(\Psi,\Sigma,\mu)$ for some compact Hausdorff Ψ , where Σ is a σ-field of subsets of Ψ . The linear span of characteristic functions of finitely many disjoint sets in Σ with union Ψ , is dense in $L^1(\Psi,\Sigma,\mu)$ by [19:III.3.8] . Let $\delta > 0$. By a similar argument (above), there exists an integer h and an h-dimensional subspace E of X^* , containing $T(S(\ell^\infty(m)))$, such that there is an invertible map U in $B(E,\ell^1(h))$ with $\| U \| = 1$, $\|U^{-1}\| \leq 1 + \delta$.

Let $T_0 = UTS : \ell^\infty(m) \xrightarrow{S} W_0 \xrightarrow{T} E \xrightarrow{U} \ell^1(h)$, and let, for $i=1,\ldots,n$, $z_i = S^{-1}x_i \in \ell^\infty(m)$. For each z^* in $\ell^1(m)$,

$$\sum_1^n |z^*(z_i)|^2 = \sum_1^n |z^*(S^{-1}x_i)|^2 = \sum_1^n |(S^{-1})^* z^*(x_i)|^2 \leq \| (S^{-1})^* z^* \|^2 .$$

If $(e_j)_1^m$ is the usual basis for $\ell^\infty(m)$, and if z_{ij} is the

j^{th} coordinate of $z_i \in \ell^\infty(m)$ in this basis for each i,j, then, by taking z^* above to be the j^{th} unit vector in $\ell^1(m)$,

$$\sum_{i=1}^{n} |z_{ij}|^2 \leq (1+\varepsilon)^2 \quad \text{for } j=1,\ldots,m . \qquad \ldots\ldots(1)$$

Now let $(f_k)_1^h$ be the standard basis in $\ell^1(h)$, and define a_{jk} for $j=1,\ldots,m$, $k=1,\ldots,h$ by

$$T_0 e_j = \sum_{k=1}^{h} a_{jk} f_k \qquad (j=1,\ldots,m) .$$

Then

$$T_0 z_i = T_0\Big(\sum_{j=1}^{m} z_{ij} e_j\Big) = \sum_{j=1}^{m} z_{ij}\Big(\sum_{k=1}^{h} a_{jk} f_k\Big) = \sum_{k=1}^{h} \Big(\sum_{j=1}^{m} z_{ij} a_{jk}\Big) f_k$$

for $i=1,\ldots,n$, and hence

$$\|T_0 z_i\|_{\ell^1(h)} = \sum_{k=1}^{h} \Big|\sum_{j=1}^{m} z_{ij} a_{jk}\Big| \qquad (i=1,\ldots,n) .$$

Now

$$\sum_{i=1}^{n} \|Tx_i\|^2 = \sum_{i=1}^{n} \|U^{-1} T_0 z_i\|^2 \leq \|U^{-1}\|^2 \sum_{i=1}^{n} \|T_0 z_i\|^2$$

$$\leq (1+\delta)^2 \sum_{i=1}^{n} \Big(\sum_{k=1}^{h} \Big|\sum_{j=1}^{m} z_{ij} a_{jk}\Big|\Big)^2 .$$

Thus, in order to show that $\pi_2(T) \leq (1+\delta)(1+\varepsilon)\, K_G \|T\|$, where $K_G \leq 2 \sinh \frac{\pi}{2}$, it suffices to prove that

$$\sum_{i=1}^{n} \Big(\sum_{k=1}^{h} \Big|\sum_{j=1}^{m} z_{ij} a_{jk}\Big|\Big)^2 \leq (1+\varepsilon)^2\, K_G^2 \|T_0\|^2 , \qquad \ldots\ldots(2)$$

since $\|T_0\| \leq \|T\|$.

Set $b_{ki} = \Big|\sum_{j=1}^{m} z_{ij} a_{jk}\Big|$ for $k=1,\ldots,h$, $i=1,\ldots,n$.

If $c = (c_1,\ldots,c_h) \in \ell^1(h)^*$ with $\|c\| = 1$, and if $t = (t_k)_1^n$ and $(s_j)_1^m$ are any 2 sequences in $\mathbb{C}$ such that $|s_j|, |t_k| \leq 1$ for all j,k, then:

$$\left|\sum_{j,k} a_{jk}c_k s_j t_k\right| = \left|(tc)\left(\sum_{j=1}^{m} T_0 s_j e_j\right)\right| \leq \|tc\| \|T_0\| \cdot \left\|\sum_{j=1}^{m} s_j e_j\right\| \leq \|T_0\| .$$

By (1) of Corollary 2.9 and (1) above,

$$\sum_{k=1}^{h} |c_k| \left(\sum_{i=1}^{n} \left|\sum_{j=1}^{m} z_{ij} a_{jk}\right|^2\right)^{\frac{1}{2}} \leq (1+\varepsilon)\cdot K_G \|T_0\| ,$$

where $K_G \leq 2 \sinh \frac{\pi}{2}$. Since this holds for all $c \in \ell^1(h)^*$ for which $\|c\| = 1$, it is also true that

$$\sum_{k=1}^{h} \left(\sum_{i=1}^{n} b_{ki}^2\right)^{\frac{1}{2}} \leq (1+\varepsilon)\cdot K_G \|T_0\| .$$

By the triangle inequality in ℓ^2 , (2) now follows.
Hence $\pi_2(T) \leq (1+\delta)(1+\varepsilon) K_G \|T\|$. Since δ and ε were arbitrary it follows that $\pi_2(T) \leq K_G \|T\|$. This finishes the proof. □

The following remarkable theorem is a generalization of Grothendieck's result.

2.11 THEOREM (Pisier [43])
Let A and B be C*-algebras, at least one of which has the metric approximation property, and let F be a bounded bilinear form on $A \times B$. Then there exist positive linear forms f and g in the unit balls of A* and B* respectively such that

$$|F(x,y)| \leq K \|F\| \{f(\frac{xx^*+x^*x}{2})\} \{g(\frac{yy^*+y^*y}{2})\}$$

for all $(x,y) \in A \times B$, where K is a universal constant ($K \leq 12$).
In particular, every bounded linear map from A into B* factors through a Hilbert space. □

3. Arens regularity

Let A be a Banach algebra, and let $\pi : A \longrightarrow A^{**}$ be the canonical injection. In [3], Arens defined 2 Banach algebra products on A^{**}, both of which extend the original product on A (under π), but which do not necessarily coincide; indeed, the algebra A is said to be Arens regular if and only if they do. A comprehensive survey of the subject is [18].

3.1 DEFINITION

Let A be a Banach algebra. For elements x,y of A, f of A^* and F,G of A^{**}, define

fx in A^* by $fx(y) = f(xy)$;

Ff in A^* by $Ff(x) = F(fx)$;

FG in A^{**} by $FG(f) = F(Gf)$.

The product FG is the first Arens product ; the above products will often be denoted by $f*x$, $F*f$, and $F*G$.

Similarly, define the second Arens product :

xf in A^* by $xf(y) = f(yx)$;

fF in A^* by $fF(x) = F(xf)$;

$F\cdot G$ in A^{**} by $F\cdot G(f) = G(fF)$.

The algebra A is Arens regular if $F\cdot G = FG$ for F,G in A^{**}.

It is clear that the elements of A^* and A^{**} defined above indeed belong to the appropriate spaces. For example, if $\alpha,\beta \in \mathbb{C}$, then $Ff(\alpha x+\beta y) = F(f(\alpha x+\beta y)) = F(\alpha fx + \beta fy) = \alpha(Ff)(x) + \beta(Ff)(y)$,

since $f(\alpha x+\beta y)(z) = f(\alpha xz + \beta yz) = (\alpha fx + \beta fy)(z)$ for z in A.
Also, $|Ff(x)| = |F(fx)| \leq \|F\|\cdot\|f\|\cdot\|x\|$, and so $Ff \in A^*$.
It is likewise easy to show that the 2 Arens products are associative; under either of them, A^{**} becomes a Banach algebra [3]. It follows from Definition 3.1 that when A is commutative, the Banach algebra A^{**} is commutative if and only if A is Arens regular.

3.2 LEMMA ([25])

Let A be a Banach algebra.

(i) The Arens products (on A^{**}) coincide if one of the factors is in $\pi(A)$.

(ii) The first Arens product is left w*-continuous; the second Arens product is right w*-continuous.

Proof:

(i) Let $a \in A$, let $G \in A^{**}$, then for h in A^*,

$$(\pi(a)\ G)(h) = \pi(a)(Gh) = Gh(a) = G(ha), \text{ and for } x \text{ in } A,$$

$$(h\pi(a))(x) = \pi(a)(xh) = (xh)(a) = h(ax) = ha(x),$$

and so $h\pi(a) = ha$. Thus, for h in A^*,

$$(\pi(a)\cdot G)(h) = G(h\pi(a)) = G(ha) = (\pi(a)\ G)(h),$$

and hence $\pi(a)\cdot G = \pi(a)\ G$. The proof of $G\cdot\pi(a) = G\,\pi(a)$ is similar.

(ii) Let $F_\mu \xrightarrow{w^*} F$ in A^{**}, then for every h in A^*,

$$(F_\mu G)(h) = F_\mu(Gh) \longrightarrow F(Gh) = (FG)(h),$$

hence $F_\mu G \xrightarrow{w^*} FG$. Similarly for the other product.$\square$

3.3 DEFINITION

Let A be a Banach algebra, let f belong to A^*; then f is said to be <u>weakly almost periodic</u> (or WAP) if the bounded linear map $a \longmapsto fa : A \longrightarrow A^*$ is weakly compact; and f is (uniformly) <u>almost periodic</u> if the same map is compact.

<u>3.4 THEOREM</u> ([18;25])

Let A be a Banach algebra. The following are equivalent:

(i) A is Arens regular;

(ii) each f in A^* is WAP;

(iii) the first Arens product is right w*-continuous;

(iv) for each pair of bounded sequences (x_m), (y_n) in A and each f in A^*, the limits

$$\lim_m \lim_n f(x_m y_n) \quad \text{and} \quad \lim_n \lim_m f(x_m y_n)$$

are equal whenever they both exist;

(v) for each F and G in A^{**} and f in A^*, there exist nets (x_μ), (y_ν) in the unit ball of A such that $F = \text{w*-}\lim_\mu \pi(x_\mu)$, $G = \text{w*-}\lim_\nu \pi(y_\nu)$, and

$$\lim_\mu \lim_\nu \pi(x_\mu)\ \pi(y_\nu)(f) = \lim_\nu \lim_\mu \pi(x_\mu)\ \pi(y_\nu)(f) .$$

<u>Proof</u>:

<u>(i)<=>(v)</u>: given F, G in A^{**} and f in A^*, by Goldstine's theorem [19:V.4.5] there exist nets (x_μ), (y_ν) in the unit ball of A with $F = \text{w*-}\lim_\mu \pi(x_\mu)$ and $G = \text{w*-}\lim_\nu \pi(y_\nu)$.

By Lemma 3.2(i),

$$\lim_\mu \lim_\nu \pi(x_\mu)\ \pi(y_\nu)(f) = \lim_\mu \lim_\nu \pi(x_\mu)\cdot\pi(y_\nu)(f) ,$$

which, by Lemma 3.2(ii), equals $\lim_\mu \pi(x_\mu)\cdot G(f)$; by the same lemma, this is $\lim_\mu \pi(x_\mu)\ G(f) = F\ G(f)$.

On the other hand, using the same lemma again,

$$\lim_\nu \lim_\mu \pi(x_\mu)\ \pi(y_\nu)(f) = \lim_\nu F\ \pi(y_\nu)(f) = \lim_\nu F\cdot\pi(y_\nu)(f) = F\cdot G(f) .$$

Since this holds for all f in A^*, (i) is equivalent to (v).

<u>(i)=>(ii)</u>: let f belong to A^*, and let $\phi(x) = fx \quad (x \in A)$. Then ϕ^{**} is a bounded linear map from A^{**} into A^{***}. If $\pi_1 : A^* \longrightarrow A^{***}$ is the natural imbedding, then, for F, G in A^{**},

$$\phi^{**}(F)(G) = F(\phi^*(G)) = F(Gf) = G(fF) = \pi_1(fF)(G)$$

since A is Arens regular, and so $\phi^{**}(A^{**}) \subset \pi_1(A^*)$. By Definition 1.21, the map ϕ is weakly compact; hence (ii).

<u>(ii)=>(v)</u>: let $F, G \in A^{**}$, $f \in A^*$; as in the proof of (i)<=>(v), $F = w^*\text{-}\lim_{\mu} \pi(x_\mu)$ and $G = w^*\text{-}\lim_{\nu} \pi(y_\nu)$ for some x_μ, y_ν in A of norm at most 1. By (ii), the map $\phi : x \longmapsto fx \quad (x \in A)$ is weakly compact; then $(fx_\mu)_\mu = (\phi(x_\mu))_\mu$ has a weakly convergent subnet (fx_{μ_κ}), say. Since $\pi(y_\nu) \xrightarrow{w^*} G$,

$$G(w\text{-}\lim_{\mu_\kappa} fx_{\mu_\kappa}) = \lim_{\mu_\kappa} G(fx_{\mu_\kappa}) = \lim_{\mu_\kappa} \lim_{\nu} \pi(y_\nu)(fx_{\mu_\kappa}) .$$

Thus $G(w\text{-}\lim_{\mu_\kappa} fx_{\mu_\kappa}) = \lim_{\mu_\kappa} \lim_{\nu} \pi(x_{\mu_\kappa}) \cdot \pi(y_\nu)(f)$; by Lemma 3.2, this equals $\lim_{\mu_\kappa} \lim_{\nu} \pi(x_{\mu_\kappa}) * \pi(y_\nu)(f)$.

Furthermore,

$$\begin{aligned} G(w\text{-}\lim_{\mu_\kappa} fx_{\mu_\kappa}) &= \lim_{\nu} \pi(y_\nu)(w\text{-}\lim_{\mu_\kappa} fx_{\mu_\kappa}) = \lim_{\nu} \lim_{\mu_\kappa} \pi(y_\nu)(fx_{\mu_\kappa}) \\ &= \lim_{\nu} \lim_{\mu_\kappa} \pi(x_{\mu_\kappa}) * \pi(y_\nu)(f) . \end{aligned}$$

Thus the 2 double limits are equal for (x_{μ_κ}) and (y_ν). The proof of (i)<=>(v) shows that whenever $\pi(x_\mu) \xrightarrow{w^*} F$ and $\pi(y_\nu) \xrightarrow{w^*} G$, the limits $\lim_{\mu} \lim_{\nu} \pi(x_\mu) * \pi(y_\nu)(f)$ and $\lim_{\nu} \lim_{\mu} \pi(x_\mu) * \pi(y_\nu)(f)$ both exist (and equal $F * G(f)$, $F \cdot G(f)$ respectively) for all f; hence (v) follows.

<u>(i)=>(iii)</u>: let $G_\mu \xrightarrow{w^*} G$, then, for f in A^*, F in A^{**},

$$F * G_\mu(f) = F \cdot G_\mu(f) = G_\mu(fF) \longrightarrow G(fF) = F \cdot G(f) = F * G(f) .$$

(iii)=>(i): for G in A^{**}, choose (y_ν) in the unit ball of A by Goldstine's theorem so that $G = w^*\text{-}\lim_\nu \pi(y_\nu)$. Then, for F in A^{**} and f in A^*, $\lim_\nu F\,\pi(y_\nu)(f) = F\,G(f)$, by (iii); but by Lemma 3.2, $\lim_\nu F\,\pi(y_\nu)(f) = \lim_\nu F\cdot\pi(y_\nu)(f) = F\cdot G(f)$, hence (i).

(ii)<=>(iv): by a theorem of Grothendieck [22; 27: Lemma 19.1], the weak closure M^{-w} of a bounded subset M of A^* is weakly compact if and only if for every pair of sequences (ϕ_m) in M and (F_n) in the unit ball of A^{**}, the limits $\lim_m \lim_n F_n(\phi_m)$ and $\lim_n \lim_m F_n(\phi_m)$ are equal whenever they both exist. If (x_m) and (y_n) are bounded sequences in A, then $M = \{f x_m : m \in N\}$ is bounded for each f in A^*; if f is WAP, then M is weakly compact, and by multiplying by a constant if necessary, the sequence $\pi(y_n)$ is in the unit ball of A^{**}. Since $\pi(y_n)(f x_m) = f(x_m y_n)$ for all m,n, (iv) follows. Similarly, (iv) implies (ii). □

3.5 NOTE

Condition (iv) is Pym's double limit criterion [44]. It is clear that the 5 conditions are also equivalent to:

(ii)' the map $x \longmapsto xf$ is weakly compact for f in A^*;

(iii)' the second Arens product is left w*-continuous.

3.6 COROLLARY

Let A be an Arens regular Banach algebra, let B be a closed subalgebra of A, and let I be a closed two-sided ideal of A. Then B and A/I are Arens regular. □

<u>3.7 THEOREM</u> (First proved by Civin and Yood [11])

If A is a C^*-algebra, then A is Arens regular.

<u>Proof</u>:

By a result of Akemann [1 : Corollary II.9] , every bounded linear map from A into A^* is weakly compact; alternatively, if A has the metric approximation property, then Theorem 2.11 shows that every map in $B(A,A^*)$ factors through a Hilbert space H , and since H is reflexive, it follows that $B(A,A^*) = W(A,A^*)$ by [35:42.2(3)] . In particular, therefore, every f in A^* is WAP , and so A is Arens regular by Theorem 3.4 .□

<u>3.8 THEOREM</u> (Young [62])

If G is a locally compact group, then $L^1(G)$ is Arens regular if and only if G is finite.□

CHAPTER II : ARENS REGULARITY OF THE PROJECTIVE TENSOR PRODUCT OF BANACH ALGEBRAS

4. A necessary condition

Let A and B be Banach algebras. As indicated by the title, the problem discussed in this chapter is to find necessary and sufficient conditions for $A\hat{\otimes}B$ to be Arens regular. The results below show that except in essentially trivial cases, the Arens regularity of $A\hat{\otimes}B$ implies that of A and B, but the converse does not hold in general.

4.1 THEOREM

Let A and B be Banach algebras such that

$$A^2 = \{ a_1 a_2 : a_1, a_2 \in A \} \neq \{0\} \neq B^2.$$

If $A\hat{\otimes}B$ is Arens regular, then A and B are Arens regular.

4.2 LEMMA

Let A and B be Banach algebras.

(i) $A\hat{\otimes}B$ is Arens regular if and only if $B\hat{\otimes}A$ is Arens regular.

(ii) If $f \in A^*$ satisfies the double limit criterion (Theorem 3.4) then so does cf $(c \in \mathbb{C})$.

Proof:

Part (i) follows from Corollary 1.18 , while part (ii) is clear from the statement of the criterion.□

4.3 LEMMA

Let A and B be Banach algebras such that $A^2 \neq \{0\} \neq B^2$, and let ϕ in $(A\hat{\otimes} B)^*$ satisfy the double limit criterion. For each element $b = b_1 b_2 \in B^2\setminus\{0\}$, the linear functional $\phi(\cdot, b) \in A^*$ satisfies the double limit criterion (in A). Similarly, for each $a = a_1 a_2 \in A^2\setminus\{0\}$, the linear functional $\phi(a, \cdot) \in B^*$ satisfies the criterion (in B).

Proof:

Let $\phi \in (A\hat{\otimes} B)^*$ as required, regarded here as a member of the space $\mathrm{Bil}(A\times B, \mathbb{C})$. Let $b = b_1 b_2 \in B^2\setminus\{0\}$, and suppose that (x_m) and (y_n) are bounded sequences in A such that there exist complex numbers μ and ν satisfying

$$\lim_m \lim_n \phi_b(x_m y_n) = \mu, \qquad \lim_n \lim_m \phi_b(x_m y_n) = \nu,$$

where $\phi_b(\cdot) = \phi(\cdot, b) \in A^*$. The criterion is satisfied by $\phi_b \in A^*$ if it can be shown that $\mu = \nu$.

Now (x_m) and (y_n) are bounded, and, for $m, n \in \mathbb{N}$,

$$\|x_m \otimes b_1\| = \|x_m\|\cdot\|b_1\| \quad , \quad \|y_n \otimes b_2\| = \|y_n\|\cdot\|b_2\| \ ;$$

hence the sequences $(x_m \otimes b_1)$ and $(y_n \otimes b_2)$ are bounded in $A\hat{\otimes} B$. Moreover,

$$\lim_m \lim_n \phi((x_m \otimes b_1)(y_n \otimes b_2)) = \lim_m \lim_n \phi(x_m y_n, b) = \mu, \text{ and}$$

$$\lim_n \lim_m \phi((x_m \otimes b_1)(y_n \otimes b_2)) = \lim_n \lim_m \phi(x_m y_n, b) = \nu .$$

Thus the two double limits both exist; since ϕ satisfies the criterion in $A\hat{\otimes} B$, it follows that $\mu = \nu$.

The same argument shows, mutatis mutandis, that $\phi(a, \cdot) \in B^*$ satisfies the double limit criterion for each $a = a_1 a_2 \in A^2\setminus\{0\}$. $\square$

Proof of Theorem 4.1:

Suppose $A^2 \neq \{0\} \neq B^2$ and $A \hat{\otimes} B$ is Arens regular. Since $B^2 \neq \{0\}$ there exist b_1 , b_2 in B such that $b_1 b_2 \neq 0$. By the Hahn-Banach theorem, there is a functional $g \in B^*$ such that $g(b_1 b_2) \neq 0$ and $\|g\| = 1$. Let $f \in A^*$, and define the bilinear map

$$\phi_{fg} : A \times B \longrightarrow \mathbb{C} \quad , \quad \phi_{fg}(a,b) = f(a)\, g(b) \qquad (a \in A,\ b \in B).$$

Then $\|\phi_{fg}\| \leq \|f\| \cdot \|g\|$, and $\phi_{fg} \in \mathrm{Bil}(A \times B,\ \mathbb{C}) \simeq (A \hat{\otimes} B)^*$. As $A \hat{\otimes} B$ is Arens regular, ϕ_{fg} satisfies the double limit criterion. Now

$$\phi_{fg}(a, b_1 b_2) = f(a)\, g(b_1 b_2) = c\, f(a) \qquad \text{for } a \in A ,$$

where $c = g(b_1 b_2) \neq 0$. By Lemma 4.3 , $\phi_{fg}(\cdot, b_1 b_2) = c\, f(\cdot) \in A^*$ satisfies the double limit criterion, and by Lemma 4.2 (ii) , so does $c^{-1}(cf) = f \in A^*$. Since f was arbitrary, A is Arens regular. By a similar argument, B is also Arens regular. □

4.4 REMARK

If, in Theorem 4.1, the Banach algebra A is such that $A^2 = \{0\}$, then it follows from the definitions (3.1) that both Arens products in A and in $A \hat{\otimes} B$ are identically zero ; hence A and $A \hat{\otimes} B$ are trivially Arens regular for B an arbitrary Banach algebra.

4.5 DEFINITION

Let $L^2[0,1]$ be the Hilbert space of (equivalence classes of) functions $x : [0,1] \longrightarrow \mathbb{C}$ such that $|x|^2$ is Lebesgue-measurable and with the inner product

$$< x , y > = \int_0^1 x(z)\, \overline{y(z)}\, dz \qquad (x,y \in L^2[0,1]).$$

Define the closed subspace $H^2 = H^2[0,1]$ of $L^2[0,1]$ by

$$H^2 = \{ x \in L^2[0,1] : \hat{x}(n) = \int_0^1 x(z)\, e^{-2\pi i n z}\, dz = 0 , \ n = -1,-2,\dots \} .$$

Let $P: L^2[0,1] \longrightarrow H^2[0,1]$ denote the projection defined by $P(x) = x_1$ if $x = x_1 + x_2$, where $x_1 \in H^2$, $x_2 \in (H^2)^{\perp}$, and where $(H^2)^{\perp} = \{ y \in L^2[0,1] : \langle x , y \rangle = 0 \text{ for all } x \in H^2 \}$.

Now, if $y(z) = e^{2\pi i k z}$ $(z \in [0,1])$ for some integer k, then

$$\hat{y}(n) = \int_0^1 e^{2\pi i (k-n) z}\, dz = \begin{cases} 0, & \text{if } n \neq k ; \\ 1, & \text{if } n = k . \end{cases} \qquad (n \in \mathbb{Z})$$

Thus $y \in H^2$ if $k \geq 0$, and $y \in (H^2)^{\perp}$ if $k<0$: for in this case,

$$\langle x , y \rangle = \int_0^1 x(z)\, e^{-2\pi i k z}\, dz = \hat{x}(k) = 0 \quad \text{for all } x \in H^2 .$$

4.6 <u>THEOREM</u> ([13])

The Banach algebra $C[0,1] \hat{\otimes} C[0,1]$ is not Arens regular.

<u>Proof</u>:

Let $V = C[0,1] \hat{\otimes} C[0,1]$. Let $f_n \in C[0,1]$ be the function $f_n(z) = e^{2\pi i n z}$ for $z \in [0,1]$, $n \in Z$, and define sequences (a_n) and (b_m) in V by: $a_n = f_n \otimes f_{-n}$, $b_m = a_{-m}$ $(n,m \in Z)$.

Since $\| a_n \| = \| f_n \| \cdot \| f_{-n} \| = 1 = \| b_m \|$ for n,m in Z, these are bounded sequences in V. Regard $C[0,1]$ as a subspace of $L^2[0,1]$; next, define ϕ by: $\phi(x,y) = \langle P_0 x , y^* \rangle$ for $x,y \in C[0,1]$, where P_0 is the restriction to $C[0,1]$ of the projection $P: L^2 \longrightarrow H^2$ described in Definition 4.5. Then ϕ is a bilinear form on $C[0,1] \times C[0,1]$, and $|\phi(x,y)| \leq \|x\| \cdot \|y\|$. Hence ϕ can be viewed as an element of V^*, and $\|\phi\| \leq 1$; in fact, $\|\phi\| = 1$.

As observed in Definition 4.5, each function f_k belongs to H^2 if

$k \geq 0$, and to $(H^2)^{\perp}$ if $k<0$.

Hence, for $k \geq 0$, $\phi(a_k) = \langle P_0 f_k , f_k \rangle = \langle f_k , f_k \rangle = 1$,

and for $k<0$, $\phi(a_k) = 0$.

Now $a_n b_m = (f_n \otimes \underline{f}_{-n})(f_{-m} \otimes f_{+m}) = f_{n-m} \otimes \underline{f}_{-n+m} = a_{n-m}$, and so

$$\phi(a_n b_m) = \phi(a_{n-m}) = \begin{cases} 1, & \text{if } n \geq m; \\ 0, & \text{if } n < m. \end{cases}$$

Hence $\lim_m \lim_n \phi(a_n b_m) = 1 \neq 0 = \lim_n \lim_m \phi(a_n b_m)$, and V is not Arens regular. □

4.7 REMARK

The algebra $C[0,1]$ is a C*-algebra and therefore Arens regular, by Theorem 3.7. It seems difficult to generalize the proof of Theorem 4.6 in order to show that $C(\Omega) \hat{\otimes} C(\Omega)$ is not Arens regular when Ω is a compact Hausdorff space other than a closed interval of the real line. A different approach will be used subsequently; in the next section, certain sufficient conditions for the Arens regularity of $A \hat{\otimes} B$ will be examined.

5. Completely continuous algebras

5.1 DEFINITION

A Banach algebra A is said to be <u>completely continuous</u> (or <u>CC</u>) if the left multiplication $L_x : a \longmapsto xa : A \longrightarrow A$ and the right multiplication $R_x : a \longmapsto ax : A \longrightarrow A$ are both compact for each x in A . Similarly, A is said to be <u>weakly completely</u>

continuous (or WCC) if L_x and R_x are weakly compact for each x in A; and A is said to be compact if the map $L_x R_x$ is compact for each $x \in A$. Algebras satisfying some of these conditions were studied by Kaplansky [32] and Alexander [2], amongst others.

5.2 REMARK

A CC algebra is clearly compact, and is also WCC. It is easily checked that if A is CC [resp.compact], then every closed subalgebra of A and every quotient A/I by a closed two-sided ideal I of A is also CC [resp.compact].

5.3 EXAMPLES

(i) Let Γ be a set, and let $c_0(\Gamma)$ be the Banach algebra of maps $f:\Gamma \longrightarrow \mathbb{C}$ that vanish at infinity; that is, for every $\varepsilon > 0$, $|f(x)| < \varepsilon$ for all but finitely many x's in Γ. Then $c_0(\Gamma)$ is a completely continuous algebra [33].

Similarly, if $(A_\gamma)_{\gamma\in\Gamma}$ is a family of C*-algebras, define the restricted product of the A_γ's by

$$\Pi_0(A_\gamma) = \{ (x_\gamma) \in \Pi(A_\gamma) : \forall \varepsilon > 0, \|x_\gamma\| < \varepsilon \text{ for all but finitely many } \gamma\text{'s, where } \Pi \text{ is the Cartesian product} \}.$$

When endowed with pointwise operations and the supremum norm, this product becomes a C*-algebra; the algebra $c_0(\Gamma)$ above is a restricted product of one-dimensional C*-algebras. (Here the involution is given by complex conjugation.)

(ii) If G is a compact group, then $L^1(G)$ is a CC algebra; this was shown by Segal [53:Theorem 1.11].

By Theorem 3.8, $L^1(G)$, for G an infinite compact group, thus provides an example of a completely continuous (and hence WCC and compact) Banach algebra which is not Arens regular. The C*-algebra $B(H)$, where H is a Hilbert space, is of course Arens regular (3.7) but neither WCC nor compact.

(iii) If X is a Banach space, then $K(X)$, the algebra of compact operators on X, is compact [9 :33.10]. However, $K(X)$ is not CC if X is infinite [8]. As shown in [6], if H is a Hilbert space, then $K(H)$ is a WCC algebra. Thus, in general, a compact algebra is not completely continuous, and a WCC algebra is not CC either.

5.4 DEFINITION

Let E be a subset of a Banach space X. The absolutely convex hull of E, $|co|(E)$, is defined by

$$|co|(E) = \{ \sum_1^n \alpha_k x_k \; : \; \alpha_k \in \mathbb{C},\; x_k \in E \;,\; \sum_1^n |\alpha_k| \leq 1 \;,\; n \in \mathbb{N} \} .$$

The following result was first proved by Holub [30]; the proof here is due to Köthe [35:44.6(1)].

5.5 THEOREM

Let A_1, A_2, B_1, B_2 be Banach spaces, and let $T_i : A_i \longrightarrow B_i$ be bounded linear maps for i=1,2. If T_1 and T_2 are compact, then $T_1 \hat{\otimes} T_2 : A_1 \hat{\otimes} A_2 \longrightarrow B_1 \hat{\otimes} B_2$ is compact (Definition 1.22).

Proof:

Since T_1 and T_2 are compact, there are sets $U_1 \subset A_1$, $U_2 \subset A_2$ containing neighbourhoods of 0, and compact subsets $C_1 = T_1(U_1)^-$ of B_1, $C_2 = T_2(U_2)^-$ of B_2. Then $C_1 \times C_2$ is compact in $B_1 \times B_2$.

The canonical map $(b_1,b_2) \longmapsto b_1 \otimes b_2 : B_1 \times B_2 \longrightarrow B_1 \hat{\otimes} B_2$ is continuous, and so $C_1 \otimes C_2$ is compact in $B_1 \hat{\otimes} B_2$. Now [34:20.6(3)] shows that $(|co|(C_1 \otimes C_2))^-$ is compact if and only if it is complete; but this is a closed subset of the Banach space $B_1 \hat{\otimes} B_2$, and hence is complete and compact. Since

$$((T_1 \hat{\otimes} T_2)(|co|(U_1 \otimes U_2)))^- \subset (|co|(T_1(U_1) \otimes T_2(U_2)))^- \subset (|co|(C_1 \otimes C_2))^-,$$

it follows that $((T_1 \hat{\otimes} T_2)(|co|(U_1 \otimes U_2)))^-$ is compact. Finally, $|co|(U_1 \otimes U_2)$ contains a neighbourhood of 0 in $A_1 \hat{\otimes} A_2$, and so $T_1 \hat{\otimes} T_2$ is compact.□

5.6 REMARK

If T_1 and T_2 above are only weakly compact, then $T_1 \hat{\otimes} T_2$ is not weakly compact in general [29 : last remark in §3].

5.7 LEMMA ([9 :33.12])

If A is a Banach algebra, and if E is a dense subset of A such that $b \longmapsto aba : A \longrightarrow A$ is compact for each $a \in E$, then A is a compact algebra. Similarly, if the left and the right multiplication by elements of E in A are compact, then A is CC.

5.8 PROPOSITION

If A and B are compact [CC] Banach algebras, then $A \hat{\otimes} B$ is a compact [CC] Banach algebra.

Proof:

Let A and B be compact. Let $\sum_1^n a_k \otimes b_k \in A \hat{\otimes} B$; by hypothesis, the following maps are compact:

$$S : x \longmapsto \sum_1^n a_k x a_k : A \longrightarrow A\ , \quad T : y \longmapsto \sum_1^n b_k y b_k : B \longrightarrow B\ .$$

By Theorem 5.5, $S \hat{\otimes} T : A \hat{\otimes} B \longrightarrow A \hat{\otimes} B$ is compact, where

$$(S \hat{\otimes} T)(\sum_1^m x_j \otimes y_j) = (\sum_1^n a_k \otimes b_k)(\sum_1^m x_j \otimes y_j)(\sum_1^n a_k \otimes b_k)\ ;\ x_j \in A, y_j \in B;$$

hence, by linearity and continuity,

$$(S \hat{\otimes} T)(v) = (\sum_1^n a_k \otimes b_k)\, v\, (\sum_1^n a_k \otimes b_k) \quad \text{for } v \text{ in } A \hat{\otimes} B\ .$$

Now let $E = \{ \sum_1^n a_k \otimes b_k : a_k \in A,\ b_k \in B,\ n \in \mathbb{N} \}$; then E is dense in $A \hat{\otimes} B$, and the map $v \longmapsto uvu : A \hat{\otimes} B \longrightarrow A \hat{\otimes} B$ is compact for each u in E , as shown above. By Lemma 5.7 , $A \hat{\otimes} B$ is a compact Banach algebra. The case when A and B are CC is treated in the same manner.□

The following characterization of compact C*-algebras is included in the work of Alexander [2] and Berglund [6].

5.9 THEOREM

Let A be a C*-algebra. The following are equivalent:

(i) A is compact;

(ii) A is a C*-subalgebra of $K(H)$ for some Hilbert space H ;

(iii) A is a WCC algebra;

(iv) $A = \Pi_0(A_\gamma)$, where each A_γ is an algebra of compact operators on a Hilbert space.

Also, A is CC if and only if $A = \Pi_0(A_\gamma)$, where each A_γ is a finite-dimensional C*-algebra.□

5.10 DEFINITION

If A is a Banach algebra, let $A^* * A = \{ f * a : f \in A^*\ , a \in A \}$, where

$f*a(b) = f(ab)$ for $b \in A$, the first step in defining the Arens product (3.1). Similarly, let $A \cdot A^* = \{ a \cdot f : a \in A, f \in A^* \}$, where $a \cdot f(b) = f(ba)$ for b in A. Observe that A^* is a right A-module under $*$ (1.27), and in fact a Banach module: $\|f*a\| \leq \|f\| \cdot \|a\|$. For $a_1, a_2 \in A$ and $f \in A^*$, it is clear that $f*(a_1 a_2) = (f*a_1)*a_2$.

5.11 PROPOSITION

Let A be a Banach algebra such that $A^* * A = A^*$ and the left multiplication in A weakly compact. Then A is Arens regular. Similarly, the same conclusion holds if $A \cdot A^* = A^*$ and the right multiplication weakly compact.

In particular, if A is WCC, and $A^* * A = A^*$ or $A \cdot A^* = A^*$, then A is Arens regular.

Proof:

Suppose $A^* * A = A^*$, and let $f \in A^*$. There exist $g \in A^*$ and $a \in A$ such that $f = g*a$. Hence, for all $b \in A$, $f*b = (g*a)*b = g*(ab)$. Consider the map $b \longmapsto f*b : A \longrightarrow A^*$; as $f*b = g*(ab)$, this map factors: $b \longmapsto ab \longmapsto g*(ab) = f*b$, where the left multiplication by a is weakly compact by hypothesis, and the second map is linear and bounded (3.3). Hence $b \longmapsto f*b : A \longrightarrow A^*$ is weakly compact, and so f is weakly almost periodic. Since $f \in A^*$ was arbitrary, A is Arens regular by Theorem 3.4.

The second part follows in a similar way, using Note 3.5. $\square$

5.12 REMARKS

(i) If A is a completely continuous Banach algebra and $A^* * A = A^*$, then the proof above shows that every f in A^* is (uniformly) almost periodic; that is, $b \longmapsto f*b$ is compact (Definition 3.3).

(ii) If G is an infinite compact group, then $L^1(G)$ is CC (Example 5.3(ii)), and so the above proposition shows that $L^1(G)^{**}L^1(G) \neq L^1(G)^*$ (note here that $*$ denotes the Arens product rather than the convolution) ; for otherwise $L^1(G)$ would be Arens regular, contradicting Theorem 3.8. Also, $L^1(G)\cdot L^1(G)^* \neq L^1(G)^*$.

5.13 THEOREM

Let A and B be Banach algebras with bounded right approximate identities, and suppose the following conditions are satisfied:

(i) $A^{**}A = A^*$, $B^{**}B = B^*$;

(ii) the left multiplication in A is compact;

(iii) A^* or B^* has the approximation property.

Then $V^{**}V \simeq K(A,B^*)$, where $V = A\hat{\otimes} B$.

If, in addition,

(iv) the left multiplication in B is compact, and

(v) every bounded linear map from A into B^* is compact,

then every element of V^* is (uniformly) almost periodic; hence $A\hat{\otimes} B$ is Arens regular.

5.14 NOTE

In view of Proposition 5.11, conditions (i) and (ii) of the theorem imply that A is Arens regular, and (i) and (iv) that B is Arens regular. Also, the theorem remains valid if the second Arens product is used in (i), and the other conditions appropriately modified.

Proof of Theorem 5.13:

By the standard identification, $V^* \simeq B(A,B^*)$.

Let F be a finite-rank operator in $K(A,B^*)$. There exist $(f_j) \in A^*$, $(g_j) \in B^*$ for $j = 1,2,\ldots,n$ such that $F = \sum_1^n f_j \otimes g_j$, where $(f_j \otimes g_j)(a) = f_j(a) g_j \in B^*$ for $a \in A$ (Definition 1.11).

By (5.10), A^* is a right A-module, and by hypothesis A has a bounded right approximate identity; by (i), $A^* * A = A^*$, and so by Corollary 1.30, there exist elements $h_1,\ldots,h_n$ of A^*, c of A, such that $f_j = h_j * c$ $(j=1,\ldots,n)$. Similarly, there exist $k_j \in B^*$, $d \in B$, such that $g_j = k_j * d$ $(j=1,\ldots,n)$. Define $S \in K(A,B^*)$ by $S = \sum_1^n h_j \otimes k_j$. For $x \otimes y$ in $A \hat{\otimes} B$,

$$\begin{aligned}(S*(c\otimes d))(x\otimes y) &= S(cx \otimes dy) = \sum_1^n h_j(cx)\ k_j(dy) = \\ &= \sum_1^n (h_j * c)(x)\ (k_j * d)(y) = \sum_1^n f_j(x)\ g_j(y) = \\ &= F(x \otimes y).\end{aligned}$$

By linearity and continuity, $F = S*(c\otimes d) \in K(A,B^*)*(A\hat{\otimes} B) \subset V^* * V$. Thus the finite-rank operators from A into B^* belong to $V^* * V$.

Since A and B have bounded right approximate identities, so does $A\hat{\otimes} B$, by Proposition 1.25; Theorem 1.29 implies that $V^* * V$ is $\|\cdot\|_{V^*}$-closed in V^*. By (iii), A^* or B^* has the approximation property; hence the $\|\cdot\|_{V^*}$-limits of finite-rank operators from A into B^* are precisely the compact operators from A into B^*. Therefore, $K(A,B^*) \subset V^* * V$.

For the reverse inclusion, let $f \in V^* * V$ be of the form $f = g * (\sum_1^m x_j \otimes y_j)$ for some $g \in V^*$, $x_1,\ldots,x_m \in A$, $y_1,\ldots,y_m \in B$.

Let (a_n) be a bounded sequence in A. For $b \in B$, $n \in \mathbb{N}$,

$f(a_n)(b) = (g * \sum_1^m x_j \otimes y_j)(a_n \otimes b) = \sum_{j=1}^m g(x_j a_n)(y_j b) = (\sum_1^m g(x_j a_n) * y_j)(b)$,

where now $g(x_j a_n) * y_j \in B^* * B$ for $j = 1, \ldots, m$. Thus

$$f((a_n)_n) = (\sum_{j=1}^m g(x_j a_n) * y_j)_n \subset B^* . \qquad \ldots\ldots(1)$$

By (ii), the closure of $(x_j a_n)_n$ is compact in A for each j; that is, $(x_j a_n)_n$ is relatively compact in A. Since $g \in B(A, B^*)$ is continuous, $g((x_j a_n)_n) = (g(x_j a_n))_n$ is relatively compact in B^* for each j. Finally, the map $h \longmapsto h * y_j : B^* \longrightarrow B^*$ $(j = 1, \ldots, m)$ is continuous, and hence $(\sum_{j=1}^m g(x_j a_n) * y_j)_n$ is relatively compact in B^*. By (1), $f \in K(A, B^*)$.

If now f is an arbitrary element of $V^* * V$, then, since $v \longmapsto g * v : V \longrightarrow V^*$ is continuous, f is a $\|\cdot\|_{V^*}$ - limit of compact maps of the form $g * (\sum_1^m x_j \otimes y_j)$ as above; hence $f \in K(A, B^*)$.

If, in addition, conditions (iv) and (v) hold, then the left multiplication in V is compact, and so, by Proposition 5.11 and Remark 5.12(i), every element of V^* is almost periodic; for (v) implies that $V^* * V = V^*$. □

5.15 LEMMA ([59 : Theorem 2.11])

If A is a C^*-algebra, then $A \cdot A^* = A^* = A^* * A$. □

5.16 NOTE

If A and B are C^*-algebras, then they have bounded (right) approximate identities by Definition 1.24, and by Lemma 5.15, condition (i) of Theorem 5.13 is fulfilled. This theorem will now be applied to C*-algebras.

5.17 LEMMA (For $X = \ell^1$: Schur [52], Banach [5:p.137])
Let $(E_i)_1^\infty$ be a sequence of finite-dimensional Banach spaces, and let $X = \Pi_{\ell^1}(E_i)$ be the Banach space of sequences $x = (x_i)$, $x_i \in E_i$ for each i, such that $\|x\| = \sum_1^\infty \|x_i\| < \infty$. Then the compact and the weakly compact subsets of X coincide.

Proof:
Let M be a weakly compact subset of X. Since each E_i is finite-dimensional and hence separable, the space X is also separable. By [34:21.3(5)], the dual X^* is weakly sequentially separable; in other words, there exists a countable weakly dense subset $W = \{w_1, w_2, \ldots\}$ of X^* such that every element of X^* is the weak limit of a sequence in W.

For m in M and $n=1,2,\ldots$, let $f_n(m) = w_n(m)$; that is, each $f_n : M \longrightarrow \mathbb{C}$ is the restriction of w_n to M. By the definition of the weak topology, each f_n is weakly continuous. If $f_n(m_1) = f_n(m_2)$ for all n, then $m_1 = m_2$: for suppose $f \in X^*$, then there is a sequence (w_k) in W such that $f = w\text{-}\lim_k w_k$; in particular, $f(m_1) = \lim_k w_k(m_1) = \lim_k w_k(m_2) = f(m_2)$, and since f in X^* was arbitrary, it follows that $m_1 = m_2$.

Thus (f_n) is a sequence of weakly continuous functions on the weakly compact set M, which separates the points of M. Then M is metrizable in the weak topology [50: 3.8]; since M is metric and compact in the weak topology, every sequence in M

contains a weakly convergent subsequence [34 :4.5(4)] .

To prove that M is sequentially compact and therefore compact, suppose that $(z^{(n)})$ is a sequence in X such that $z^{(n)}$ is weakly convergent to 0, but $\lim_n \|z^{(n)}\| \neq 0$. By passing to a subsequence if necessary, there exists $\varepsilon > 0$ such that

$$5\varepsilon < \|z^{(n)}\| = \sum_{i=1}^{\infty} \|z_i^{(n)}\| \quad \text{for } n=1,2,\dots \qquad \dots\dots(1)$$

Let $N \in \mathbb{N}$ be such that $\sum_{i=N+1}^{\infty} \|z_i^{(1)}\| \leq \varepsilon$; then, by (1) ,

$$5\varepsilon < \|z^{(1)}\| \leq \sum_{i=1}^{N} \|z_i^{(1)}\| + \varepsilon \quad ,$$

and so $\sum_{i=1}^{N} \|z_i^{(1)}\| > 4\varepsilon$. $\qquad \dots\dots(2)$

Now the dual of X is given by $\Pi_{\ell^\infty}(E_i^*)$, where the norm of an element $f = (f_1, f_2, \dots)$ is $\|f\| = \sup_i \|f_i\|$, and f in X^* acts on an x in X by [34:§26.8]

$$f(x) = \sum_1^{\infty} f_i(x_i) \quad , \quad x_i \in E_i \ , \ f_i \in E_i^* \quad .$$

By the Hahn-Banach theorem, there exists $v = (v_1, \dots, v_N, 0, \dots)$ in X^* such that, by (2) ,

$$v(z_1^{(1)}, \dots, z_N^{(1)}, 0, \dots) = \sum_{i=1}^{N} \|z_i^{(1)}\| > 4\varepsilon \quad ,$$

and $\|v\| = \sup_{1 \leq i \leq N} \|v_i\| = 1$.

Hence $\sum_{i=1}^{N} v_i(z_i^{(1)}) > 4\varepsilon$, where $v_i \in E_i^*$. $\qquad \dots\dots(3)$

Now consider the Banach space $\Pi_{\ell_N^1}(E_i)$ of finite sequences $x = (x_1, \dots, x_N)$, where $x_i \in E_i$, and with the ℓ^1-norm ; this space is finite-dimensional since each E_i is finite-dimensional.

Moreover, the dual of $\Pi_{\ell^1_N}(E_i)$ is $\Pi_{\ell^\infty_N}(E_i^*)$, defined similarly.

Since the sequence $(z^{(n)}) \subset X$ converges weakly to 0 , it follows that, in particular, if elements of X^* are of the form $f = (f_1,\ldots,f_N,0,\ldots)$, then $f_1(z_1^{(n)}) + \ldots + f_N(z_N^{(n)}) = f(z^{(n)})$ converges to 0 , for all f_i in E_i^* , $i=1,\ldots,N$.
In other words, the sequence

$$(z_1^{(n)},\ldots,z_N^{(n)}) \quad \text{in} \quad \Pi_{\ell^1_N}(E_i)$$

converges weakly to 0 as $n \longrightarrow \infty$. But $\Pi_{\ell^1_N}(E_i)$ is finite-dimensional, and hence all Hausdorff linear topologies on it coincide (e.g. [14:II5, Lemma 1]); hence the sequence $(z_1^{(n)},\ldots,z_N^{(n)})$ converges to 0 in norm, or

$$\sum_{i=1}^{N} \| z_i^{(n)} \| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty . \qquad \ldots\ldots(4)$$

Now let $n_1 = 1$, let $N_1 = N$, and, by (4), choose $n_2 \in \mathbb{N}$ such that

$$\sum_{i=1}^{N_1} \| z_i^{(n_2)} \| \leq \varepsilon \quad . \qquad \ldots\ldots(5)$$

Then there exists $N_2 \in \mathbb{N}$ such that

$$\sum_{i=N_2+1}^{\infty} \| z_i^{(n_2)} \| \leq \varepsilon \quad , \qquad \ldots\ldots(6)$$

implying that $N_2 > N_1$, for otherwise, by (1), $5\varepsilon < \| z^{(n_2)} \| \leq 2\varepsilon$, which is a contradiction. Hence, by (1) ,

$$\sum_{i=1}^{N_2} \| z_i^{(n_2)} \| > 4\varepsilon \ .$$

As before, there exist v_i in E_i^* , $i=N_1+1,\ldots,N_2$, such that $\sup\{ \| v_i \| \ : i=N_1+1,\ldots,N_2 \} = 1$ and

$$\sum_{i=N_1+1}^{N_2} v_i(z_i^{(n_2)}) > 4\varepsilon \ . \qquad \ldots\ldots(7)$$

Continue this process: by (4), there exists $n_3 \in \mathbb{N}$ such that $\sum_{i=1}^{N_2} \|z_i^{(n_3)}\| \leq \varepsilon$, and hence there is $N_3 > N_2$, and so on. Thus there is a sequence $w = (v_1, \ldots, v_{N_1}, v_{N_1+1}, \ldots)$ in X^*, with $\|w\| = \sup_i \|v_i\| = 1$, and a sequence $n_1, n_2, \ldots$ in $\mathbb{N}$, such that

$$\left|\sum_{i=1}^{\infty} v_i(z_i^{(n_1)})\right| \geq \left|\sum_{i=1}^{N_1} v_i(z_i^{(n_1)})\right| - \left|\sum_{i=N_1+1}^{\infty} v_i(z_i^{(n_1)})\right|$$

$$\geq 4\varepsilon - \varepsilon = 3\varepsilon > \varepsilon \quad , \quad \text{using (3)} ;$$

$$\left|\sum_{i=1}^{\infty} v_i(z_i^{(n_2)})\right| \geq \left|\sum_{N_1+1}^{N_2}\right| - \left|\sum_{1}^{N_1}\right| - \left|\sum_{N_2+1}^{\infty}\right|$$

$$\geq 4\varepsilon - \varepsilon - \varepsilon = 2\varepsilon > \varepsilon \quad ,$$

using (7), (5) and (6), and $\|v_i\| \leq 1$ for all i; and so on.

Thus $|w(z^{(n)})| > \varepsilon$ for $n = n_1, n_2, \ldots$, where $w \in X^*$; but this contradicts the weak convergence of $(z^{(n)})$ to 0. Hence a sequence in X converges in norm if it converges weakly (and in particular X is weakly sequentially complete).

Therefore, every sequence in M contains a convergent subsequence, and hence M is compact. Conversely, if M is compact, then it is weakly compact. □

<u>5.18 PROPOSITION</u>

Let A be a separable completely continuous C*-algebra, and let X be a Banach space. Then every weakly compact bounded linear map from A into X is compact; that is, $W(A,X) = K(A,X)$.

<u>Proof</u>:

Every compact map in $B(A,X)$ is weakly compact, hence

$K(A,X) \subset W(A,X)$. Conversely, let $T \in W(A,X)$, then $T^* \in W(X^*,A^*)$ [19:VI.4] . Since A is a separable CC algebra, by Theorem 5.9 there exists a sequence (E_n) of finite-dimensional C*-algebras such that $A = \Pi_0(E_n)$, and hence $A^* = \Pi_{\ell^1}(E_n^*)$. Each E_n^* is finite-dimensional since each E_n is; by Lemma 5.17 , the compact and the weakly compact subsets of A^* coincide, and hence T^* is compact. By Schauder's theorem [19:VI.5] , the map $T : A \longrightarrow X$ is compact. Thus $W(A,X) = K(A,X)$. □

5.19 THEOREM

If A and B are completely continuous C*-algebras, and if at least one of them is separable, then $V = A \hat{\otimes} B$ is Arens regular.

Proof:

Suppose, without loss of generality, that A is separable. By Note 5.16 and the hypotheses, conditions (i),(ii) and (iv) of Theorem 5.13 are satisfied. By Proposition 5.18 , it follows that $K(A,B^*) = W(A,B^*)$, and as observed in the proof of Theorem 3.7 , it is also true that $W(A,B^*) = B(A,B^*)$, i.e. every bounded linear map from A into B^* is weakly compact. Hence the condition (v) of Theorem 5.13 is also fulfilled.

Finally, as in Proposition 5.18 , there is a sequence (E_n) of finite-dimensional C*-algebras such that $A = \Pi_0(E_n)$; since each E_n^* is finite-dimensional, it has a basis, and hence $\Pi_{\ell^1}(E_n^*)$ also has a basis. By Definition 1.31 , $A^* = \Pi_{\ell^1}(E_n^*)$ then has the approximation property. Hence the condition (iii) of Theorem 5.13 is satisfied, and so every element of V^* is almost periodic.

By Theorem 3.4 , the algebra V is Arens regular. $\square$

5.20 EXAMPLE

The Banach algebra $c_0 \hat{\otimes} c_0$ is Arens regular; every element of $(c_0 \hat{\otimes} c_0)^*$ is almost periodic.

5.21 REMARK

Let A and B be C*-algebras. It would be interesting to know whether $A \hat{\otimes} B = V$ is Arens regular if, with A separable,

(i) A is completely continuous and B compact (<=> WCC);

(ii) both A and B are compact.

In (i), Theorem 5.19 shows that $V^* * V = V^*$. However, one still needs weakly compact multiplication in V in order to deduce that V is Arens regular using Proposition 5.11 . In (ii), it seems that different methods are needed; in particular, if H is a Hilbert space, is $K(H) \hat{\otimes} K(H)$ Arens regular?

5.22 DEFINITION

Let A be a Banach algebra. A <u>double centralizer</u> on A is a pair of operators (S,T) in $B(A) \times B(A)$ such that $a(Sb) = (Ta)b$ for a,b in A . The set of double centralizers on A can be made into a Banach algebra, called the <u>double centralizer algebra</u> of A , and denoted by $M(A)$ [31] . The algebra A can be imbedded isometrically in $M(A)$ by the map $a \longmapsto (L_a, R_a)$, hence A can be viewed as a closed subalgebra of its double centralizer algebra. A net $\{ e_i : i \in I \}$ in A is called a quasicentral bounded approximate identity bounded by 1 if (e_i) is a two-sided approximate identity bounded by 1 of A, such that $\|e_i b - b e_i\| \longrightarrow 0 \quad (i \in I)$ for all b in $M(A)$.

5.23 THEOREM ([55:Theorem 5])

Let A be a Banach algebra with a two-sided bounded approximate identity bounded by 1. If $A^* = A \cdot A^* + A^* * A$, then A has a quasicentral bounded approximate identity bounded by 1. □

5.24 COROLLARY

If A and B are C*-algebras, and if A is completely continuous and separable, then $V = A \hat{\otimes} B$ has a quasicentral bounded approximate identity bounded by 1.

Proof:

As observed in Remark 5.21, the proof of Theorem 5.19 shows that $V^* * V = V^*$. □

6. Commutative C*-algebras

Every unital commutative C*-algebra can be identified with $C(\Omega)$, where Ω is a compact Hausdorff space. In this section, Arens regularity of $C(\Omega) \hat{\otimes} C(\Psi)$ for Ω and Ψ compact metric spaces will be discussed. Throughout, G will be a locally compact (Hausdorff) abelian group, with normalized Haar measure.

6.1 DEFINITION

The dual group $\hat{G}$ of G is the set of those continuous maps $\gamma : G \longrightarrow \mathbb{C}$ which satisfy $|\gamma(x)| = 1$ $(x \in G)$ and $\gamma(x+y) = \gamma(x)\cdot\gamma(y)$ $(x,y \in G)$. The group operation on $\hat{G}$ is given by $(\gamma_1 + \gamma_2)(x) = \gamma_1(x)\cdot\gamma_2(x)$ $(x \in G)$. Let $A(\hat{G}) = \{\hat{f} : f \in L^1(G)\}$, where $\hat{f}$ is the Fourier (Gelfand) transform of f .

6.2 NOTE

It is known that $\hat{G}$ is a locally compact abelian group with the weak topology induced by $A(\hat{G})$, and that $\hat{\hat{G}} = G$ (Pontrjagin duality theorem) [48:Chapter I] .

6.3 DEFINITION

Let A be a commutative algebra, and let Φ_A be the set of non-zero homomorphisms from A into $\mathbb{C}$. The carrier space of A is the set Φ_A endowed with the A-topology [45 :3.13] . The algebra A is said to be semi-simple if $\cap \{\mathrm{Ker}\,\phi : \phi \in \Phi_A\} = \{0\}$.

6.4 NOTE

The (Banach) algebra $L^1(G)$ is semi-simple [48] . Thus $L^1(G)$ and $A(\hat{G}) = (L^1(G))^\wedge$ are algebraically isomorphic [9 :17.7] . The following result was proved by Gelfand in 1948 ; see [45:2.5.18] or Johnson's generalization [9 :25.9] .

6.5 PROPOSITION

Let A be a semi-simple commutative algebra, and suppose there exist norms $\|\cdot\|_1$, $\|\cdot\|_2$ on A such that $(A, \|\cdot\|_1)$ and

$(A, \|\cdot\|_2)$ are Banach algebras. Then these norms are equivalent; thus the norm topology on a semi-simple commutative Banach algebra is unique.$\square$

6.6 DEFINITION

A closed non-empty subset of a topological space is said to be perfect if it consists only of limit points. The Cantor set, denoted by D_∞, is the space of all sequences (a_n) with a_n belonging to $\{-1,1\}$ for each n, and where the topology is the product one, induced by the discrete topology on $\{-1,1\}$. The Cantor set can be written as $\prod_1^\infty (\mathbb{Z}/2\mathbb{Z})$, and is an infinite compact abelian group; it is the only perfect, totally disconnected metrizable compact space (up to homeomorphism) [26:p.100].

6.7 THEOREM (Varopoulos [61:8.1])

Let G be a compact abelian group. Then $A(G)$ is isometrically isomorphic to a closed subalgebra of $V(G) = C(G)\hat{\otimes}C(G)$.$\square$

6.8 NOTE

The map which Varopoulos uses to establish the isometric isomorphism is constructed as follows: he defines $\bar{M} : C(G) \longrightarrow C(G \times G)$ by $\bar{M}f(x,y) = f(x+y)$ $(f \in C(G),\ x,y \in G)$, and looks at $M = \bar{M}|_{A(G)}$.
He then shows that M is an isometry on $V(G) \subset C(G\times G)$, and that it is an isomorphism : $M(A(G)) \subset V(G) \subset C(G\times G)$, where $M(A(G))$ is a closed subalgebra of $V(G)$. Here $A(G) = A(\hat{\hat{G}})$, and $V(G)$ can be naturally imbedded in $C(G\times G)$ by the map $\varepsilon : V(G) \to C(G\times G)$, $\varepsilon(f\otimes g)(x,y) = f(x)g(y)$ for $f\otimes g \in V(G)$, $x,y \in G$.

6.9 COROLLARY

If G is an infinite compact abelian group, then $C(G)\hat{\otimes}C(G)$ is not Arens regular. In particular, $V(D_\infty)$ is not Arens regular.

Proof:

Since $\hat{G}$ is an abelian locally compact group, $L^1(\hat{G})$ is a commutative semi-simple algebra. As observed in Note 6.4, the algebras $L^1(\hat{G})$ and $A(G) = A(\hat{\hat{G}}) = L^1(\hat{G})^\wedge$ are isomorphic. By Theorem 6.7, $A(G)$ can be regarded as a closed subalgebra of $V(G)$, and hence $(L^1(\hat{G}), \|\cdot\|)$ is complete, where $\|\cdot\|$ comes from $V(G)$. ~~As the norm topology on $(L^1(\hat{G}), \|\cdot\|)$ is unique (6.5), it follows that $\|\cdot\|$ is equivalent to the usual norm on $L^1(\hat{G})$.~~ By Theorem 3.8, $L^1(\hat{G})$ is not Arens regular, and hence, by Corollary 3.6, the algebra $V(G)$ is not Arens regular. □

6.10 REMARK

If Ω and Ψ are finite spaces, then $C(\Omega)\hat{\otimes}C(\Psi)$ is finite-dimensional, and therefore reflexive and Arens regular.

6.11 THEOREM

Let Ω and Ψ be compact metric spaces, and suppose that each contains a perfect set. Then the Banach algebra $C(\Omega)\hat{\otimes}C(\Psi)$ is not Arens regular.

6.12 LEMMA ([37:§40])

Every perfect compact metric space contains a subspace homeomorphic to the Cantor set. If X and Y are homeomorphic compact Hausdorff spaces, then $C(X) \cong C(Y)$. □

Proof of Theorem 6.11 :

Let $P \subset \Omega$ be perfect. Since P is closed, it is compact, and so by Lemma 6.12 , there exists a closed subspace D of P which is homeomorphic to D_∞ . The closed ideal I of $C(\Omega)$, defined by $I = \{x \in C(\Omega) : x(D) = 0\}$, is such that $C(D_\infty) \cong C(D) \cong C(\Omega)/I$. As shown in Corollary 1.18 , $V(D_\infty) \cong (C(\Omega)/I) \hat{\otimes} C(D_\infty)$ is isometrically isomorphic to $(C(\Omega) \hat{\otimes} C(D_\infty))/J$ for some closed ideal J of $C(\Omega) \hat{\otimes} C(D_\infty)$. Thus $C(D_\infty) \hat{\otimes} C(D_\infty)$ is a quotient of $C(\Omega) \hat{\otimes} C(D_\infty)$, and hence, by Corollary 6.9 , $C(\Omega) \hat{\otimes} C(D_\infty)$ is not Arens regular. Similarly, $C(\Omega) \hat{\otimes} C(D_\infty)$ is a quotient of $C(\Omega) \hat{\otimes} C(\Psi)$, and therefore $C(\Omega) \hat{\otimes} C(\Psi)$ is not Arens regular.□

6.13 REMARKS

(i) Theorem 4.6 is a special case of Theorem 6.11 , where $\Omega = \Psi = [0,1]$.

(ii) Suppose Ω and Ψ are compact metric spaces which are dispersed, that is, contain no perfect sets; it would be nice to show that $C(\Omega) \hat{\otimes} C(\Psi)$ is then Arens regular, as this would provide a converse to Theorem 6.11 . If α is an ordinal, let $\Gamma(\alpha) = \{\beta : \beta \text{ ordinal}, \beta \leq \alpha\}$, and let ω be the first infinite ordinal, and ω_1 the first uncountable ordinal. From the work of Pełczyński & Semadeni [41] and Baker [4] , it is known that every dispersed compact metric space is homeomorphic to some $\Gamma(\alpha)$, where $\alpha < \omega_1$; thus such a space is countable. However, Bessaga & Pełczyński [7] have shown that for $\omega \leq \alpha \leq \beta < \omega_1$, the Banach spaces $C(\Gamma(\alpha))$ and $C(\Gamma(\beta))$ are isomorphic if and only if $\beta < \alpha^{\omega}$. In connection with Section 5, if $A = C(\Gamma(\alpha))$ and $B = C(\Gamma(\beta))$ for some ordinals $\alpha,\beta < \omega_1$, then $A^* \simeq B^* \simeq \ell^1(\aleph_0)$; as A and B are C^*-algebras, it follows as before that $V^{**}V = V^*$, where $V = A\hat{\otimes} B$. This is not sufficient to ensure the Arens regularity of V; one still needs weakly compact multiplication in V.

CHAPTER III : REPRESENTATIONS OF $C(\Omega)\hat{\otimes}C(\Omega)$

7. A representation on $B(H)$

Let Ω be a compact Hausdorff space. The aim of this section is to construct a Hilbert space H , a natural representation θ of $C(\Omega)\hat{\otimes}C(\Omega)$ on $B(H)$, and to show that θ is bicontinuous.

7.1 DEFINITION

Let $(H_i)_{i\in I}$ be a family of Hilbert spaces. For each i in I , let $\pi_i(x)$ denote the i^{th} coordinate of the element x of the Cartesian product $\prod_{i\in I} H_i$. Define the set $H = \bigoplus_{i\in I} H_i$ by

$$H = \{ x \in \prod_{i\in I} H_i : \sum_i \|\pi_i(x)\|^2 < \infty \} .$$

With the linear space operations defined pointwise and the inner product given by

$$\langle x , y \rangle_H = \sum_i \langle \pi_i(x) , \pi_i(y) \rangle_{H_i} \qquad (x,y \in H) ,$$

the space H becomes a Hilbert space, with the norm

$$\| x \|^2 = \langle x , x \rangle \qquad (x \in H) \qquad [50: 12.41] .$$

If, for each i in I , there exists a (continuous) $*$-representation $\phi_i : C(\Omega) \longrightarrow B(H_i)$, then the sum of the ϕ_i's , denoted here by $\bigoplus_{i\in I}\phi_i$, is a $*$-representation $\phi : C(\Omega) \longrightarrow B(\bigoplus_{i\in I} H_i)$, where $\pi_i(\phi(x)h) = \phi_i(x)(\pi_i(h))$ for h in $\bigoplus_i H_i = H$ [16: 2.2.3].

7.2 LEMMA

Let Ω be a compact Hausdorff space, and let $V = V(\Omega) = C(\Omega)\,\hat{\otimes}\,C(\Omega)$. For each element $u \neq 0$ of V, there exists a separable Hilbert space H_u, and a *-representation $\phi_u : C(\Omega) \longrightarrow B(H_u)$.

If the map $\theta_u : V \longrightarrow B(B(H_u))$ is defined by

$$\theta_u(x \otimes y)\,T = \phi_u(x)\,T\,\phi_u(y) \qquad \text{for } x, y \text{ in } C(\Omega)\,,\ T \text{ in } B(H_u),$$

then this map is a unital homomorphism, $\|\theta_u\| \leq 1$, and there is an absolute constant $K < 5$ (independent of u) such that $\|\theta_u(u)\| \geq K^{-1}\|u\|$.

Proof:

Fix u in V, $u \neq 0$; by the Hahn-Banach theorem, there exists F in V^* such that $F(u) = \|u\|$ and $\|F\| = 1$. Using the usual identification (Proposition 1.19), the functional F can be regarded as an element of $\mathrm{Bil}\,(C(\Omega)\times C(\Omega)\,,\,\mathbb{C}\,)$. By Grothendieck's theorem (Theorem 2.1), there exists a probability measure λ on Ω, and a bounded bilinear map $F' : L^2(\Omega,\lambda)\times L^2(\Omega,\lambda) \longrightarrow \mathbb{C}$ extending F, such that $|F'(x,y)| \leq 2K_G^2\|F\|\cdot\|x\|\cdot\|y\|$ for x, y in $L^2(\Omega,\lambda)$; as observed in (2.1), the space $C(\Omega)$ is here regarded as a dense subspace of $L^2(\Omega,\lambda)$. Hence $\|F'\|_b \leq K$, where K is an absolute constant, $K < 5$; take $K = 2K_G^2$.

Let $H_u = L^2(\Omega,\lambda)$; then H_u is a separable Hilbert space. In order to define ϕ_u, consider the left multiplication by an element x of $C(\Omega)$: $L_x(y) = xy$ $(y \in C(\Omega))$. The map L_x, from $C(\Omega)$ into $C(\Omega) \subset L^2(\Omega,\lambda)$, is clearly continuous in the $L^2(\Omega,\lambda)$-norm. Since $C(\Omega)$ is dense in $L^2(\Omega,\lambda)$, this multiplication can be extended to a map $\phi_u(x) \in B(H_u)$ by linearity and continuity. Now define $\phi_u : C(\Omega) \longrightarrow B(H_u)$ by $\phi_u : x \longmapsto \phi_u(x)$.

Alternatively, since λ is a probability measure on Ω , it gives rise to a positive linear form f_λ on $C(\Omega)$, such that

$$f_\lambda(x) = \int_\Omega x \, d\lambda \quad \text{for } x \text{ in } C(\Omega) .$$

By the Gelfand-Naimark-Segal construction [16:2.4.4] , this f_λ induces a representation ϕ_u of $C(\Omega)$ on a Hilbert space ; it is easily seen from the construction that the Hilbert space is then precisely $L^2(\Omega,\lambda)$, and that $\phi_u(x) = L_x$ for x in $C(\Omega)$. Moreover, $\|\phi_u(x)\| \leq \|x\|_{C(\Omega)}$ for x in $C(\Omega)$.

Now define, for x,y in $C(\Omega)$, the map $\theta_u(x,y) : B(H_u) \longrightarrow B(H_u)$ by

$$\theta_u(x,y)(S) = \phi_u(x) \, S \, \phi_u(y) \quad \text{for } S \text{ in } B(H_u) .$$

Clearly, $\theta_u(x,y)$ is well-defined and linear. For S in $B(H_u)$, from $\|\theta_u(x,y)S\| \leq \|x\|\cdot\|y\|\cdot\|S\|$, it follows that $\|\theta_u(x,y)\| \leq \|x\|\cdot\|y\|$, and hence $\theta_u(x,y) \in B(B(H_u))$. From now until the end of the proof, the suffix "u" will be dropped from H_u , ϕ_u and θ_u ; thus $H = H_u$, $\phi = \phi_u$, and $\theta = \theta_u$.

The map $\theta : C(\Omega)\times C(\Omega) \longrightarrow B(B(H))$, $(x,y) \longmapsto \theta(x,y)$, is then well-defined, bilinear (from the linearity of ϕ), and bounded: as shown above, $\|\theta\| \leq 1$. Thus θ can be regarded as an element of $B(V,B(B(H)))$, of norm at most 1 . If $S \in B(H)$, $x,y,z,w \in C(\Omega)$, then $\theta(x\otimes y)\cdot\theta(z\otimes w)\,(S) = \theta(x\otimes y)(\phi(z)S\phi(w)) = \phi(x)\phi(z)S\phi(w)\phi(y)$

$$= \phi(xz)\,S\,\phi(yw) = \theta(xz\otimes yw)(S) = \theta((x\otimes y)(z\otimes w))(S) ,$$

since ϕ is a homomorphism and $C(\Omega)$ is commutative. By linearity and continuity, it follows that $\theta(vw) = \theta(v)\cdot\theta(w)$ for all v,w in V, and so θ is a homomorphism.

Moreover, $\theta(1\otimes 1)(S) = \phi(1)\cdot S\cdot\phi(1) = S$ for S in $B(H)$, since $\phi(1) = \mathrm{id}_{B(H)}$; hence $\theta(1\otimes 1) = \mathrm{id}_{B(B(H))}$, and θ is unital.

Let H_* denote the Banach space of conjugate-linear functionals on the Hilbert space $H = L^2(\Omega,\lambda)$; that is, $f\in H_*$ if and only if $f : H \longrightarrow \mathbb{C}$ is continuous and $f(\alpha h+\beta k) = \bar{\alpha}f(h)+\bar{\beta}f(k)$ for α,β in $\mathbb{C}$, h,k in H, and the norm given by $\|f\| = \sup\limits_{\|h\|\leq 1}|f(h)|$. Denote by τ the conjugate isometric isomorphism between $\mathrm{Bil}(H\times H,\mathbb{C})$ and $B(H,H_*)$; thus, if $x,y\in H$, $G\in \mathrm{Bil}(H\times H,\mathbb{C})$, then

$$(\tau G)y\in H_* \quad , \quad (\tau G)y(x) = \overline{G(x,y)} \quad .$$

Now recall F' in $\mathrm{Bil}(H\times H,\mathbb{C})$ from the beginning of the proof. For each y in H, the element $(\tau F')y\in H_*$; by the Riesz representation theorem, there is a unique Ty in H such that

$$(\tau F')y(x) = \langle x^*, Ty\rangle \qquad (x\in H) ,$$

where $x^*(t) = \overline{x(t)}$ for t in Ω . Then $T : y \longmapsto Ty : H \longrightarrow H$ is linear and continuous, since, for y in H ,

$$\|Ty\| = \|(\tau F')y\| \leq \|\tau F'\|\cdot\|y\| = \|F'\|\cdot\|y\| \leq K\cdot\|F\|\cdot\|y\| \quad ;$$

hence $T\in B(H)$ and $\|T\| \leq K\cdot\|F\| = K$. In view of the definition of τ , it follows that

$$\langle x^*, Ty\rangle = (\tau F')y(x) = \overline{F'(x,y)} \qquad (x,y\in H) .$$

This particular T is now used to show that $\|\theta(u)\| \geq K^{-1}\|u\|$.

For each finite tensor $v = \sum_1^n x_j \otimes y_j$ in V,

$$< \theta(v)\, T(1)\ ,\ 1 > = \sum_1^n < \phi(x_j)\, T(\phi(y_j)(1))\ ,\ 1 >$$

$$= \sum_1^n < L_{x_j} T(L_{y_j} 1)\ ,\ 1 > = \sum_1^n < Ty_j\ ,\ x_j^* >$$

$$= \sum_1^n F'(x_j, y_j) = \sum_1^n F(x_j \otimes y_j) = F(v)\ .$$

Hence $|F(v)| \leq \| \theta(v)\, T(1) \| \cdot \| 1 \| \leq \| \theta(v) T \| \leq \| \theta(v) \| \cdot \| T \|$

$$\leq K \cdot \| \theta(v) \|\ .$$

Now if $u \in V$ chosen at the start of the proof is a finite tensor, then $\|u\| = F(u) \leq K \cdot \| \theta(u) \|$. Otherwise, given $\varepsilon > 0$, there exists a finite tensor v in V such that $\| u - v \| < \varepsilon$; since θ , $< \cdot , \cdot >$ and F are continuous, it follows that

$$|\ \| u \|_{\wedge} - < \theta(u)\, T(1)\ ,\ 1 >| \leq \varepsilon(K \cdot \| \theta \| + \| F \|\ ) \leq \varepsilon(K+1) \leq 6\,\varepsilon\ .$$

As ε was arbitrary, $\| u \| \leq K \cdot \| \theta(u) \|$. Thus $\theta = \theta_u$ is a unital, norm-reducing (that is, $\| \theta \| \leq 1$) homomorphism, and

$$\| \theta_u(u) \| \geq K^{-1} \| u \|\ . \square$$

<u>7.3 COROLLARY</u> of the proof

Let Ω be a compact Hausdorff space, let F be a bounded bilinear form from $C(\Omega) \times C(\Omega)$ into $\mathbb{C}$. Then there exists a separable Hilbert space H , a *-representation ϕ of $C(\Omega)$ on H , and elements $T \in B(H)$, $h \in H$ with $\| h \| = 1$, such that

$$F(x,y) = < \phi(x) \cdot T \cdot \phi(y)\ h\ ,\ h >$$

for all x,y in $C(\Omega)$. Moreover, $\| F \| \leq \| T \| \leq K\, \| F \|$ for an absolute constant $K < 5$. $\square$

7.4 THEOREM

Let Ω be a compact Hausdorff space, let $V = V(\Omega) = C(\Omega)\hat{\otimes}C(\Omega)$. There exists a Hilbert space H and a $*$-representation ϕ of $C(\Omega)$ on H. If $\theta : V \longrightarrow B(B(H))$ is defined by

$$\theta(x\otimes y)T = \phi(x)T\phi(y) \quad \text{for } x,y \text{ in } C(\Omega), \ T \text{ in } B(H),$$

then θ is a unital, norm-reducing monomorphism such that $\|\theta^{-1}\| \leq K$ for an absolute constant $K < 5$.

Proof:

Lemma 7.2 shows that for each $u \neq 0$ in V, there exists a $*$-representation ϕ_u of $C(\Omega)$ on a Hilbert space H_u. Let ϕ be $\oplus\,\phi_u$, and let $H = \oplus\, H_u$, summed over all $u \neq 0$ in V; then $\phi : C(\Omega) \longrightarrow B(H)$ is a $*$-representation of $C(\Omega)$ on the Hilbert space H (Definition 7.1).

Define $\theta : V \longrightarrow B(B(H))$ by $\theta(x\otimes y)T = \phi(x)\cdot T\cdot\phi(y)$ for x,y in $C(\Omega)$ and T in $B(H)$. As in Lemma 7.2, it is easy to check that θ is a unital homomorphism, and that $\|\theta\| \leq 1$; all that remains to be shown is $\|\theta(u)\| \geq K^{-1}\|u\|$ for all u in V.

First of all, fix $u \neq 0$ in V. For $x \in H_u$, define $e_u(x)$ in $H = \bigoplus_{\substack{w\in V\\ w\neq 0}} H_w$ by

$$\pi_w(e_u(x)) = \begin{cases} x, & \text{if } w=u; \\ 0, & \text{otherwise}. \end{cases}$$

Then $e_u(x) \in H$ has x as the u^{th} coordinate and 0 elsewhere, $\pi_u(e_u(x)) = x$, and $\|e_u(x)\|_H = \|x\|_{H_u}$. Thus e_u is an isometry in $B(H_u, H)$.

Given S in $B(H_u)$, define $\tilde{S}$ in $B(H)$ by $\tilde{S} = e_u \circ S \circ \pi_u$.

Then $\tilde{S}$ is linear, and $\|\tilde{S}\| = \|e_u \circ S \circ \pi_u\| \leq \|S\|$. Also, since $\pi_u \circ e_u = id_{H_u}$, it follows that $S = \pi_u \circ \tilde{S} \circ e_u$, and so $\|\tilde{S}\| = \|S\|$.

As observed in Definition 7.1 ,

$$\pi_u(\phi(x)h) = \phi_u(x)(\pi_u h) \qquad \text{for all } h \text{ in } H ,\ x \text{ in } C(\Omega) .$$

Hence, if $S \in B(H_u)$, $x,y \in C(\Omega)$, and $h \in H$, then

$$\pi_u(\phi(x)\tilde{S}\phi(y)h) = \phi_u(x)\cdot\pi_u(\tilde{S}\phi(y)h) = \phi_u(x)\cdot S(\pi_u(\phi(y)h))$$
$$= \phi_u(x)\cdot S\cdot\phi_u(y)(\pi_u h) .$$

Thus, for all S in $B(H_u)$, x,y in $C(\Omega)$, h in H ,

$$\pi_u(\theta(x\otimes y)\tilde{S}h) = \theta_u(x\otimes y)S(\pi_u h) , \text{ with } \theta_u \text{ as in (7.2)} .$$

By linearity and continuity, for v in V , S in $B(H_u)$, h in H ,

$$\pi_u(\theta(v)\tilde{S}h) = \theta_u(v)S(\pi_u h) \qquad \text{.....(1)}$$

Therefore, for v in V , S in $B(H_u)$,

$$\|\theta(v)\tilde{S}\| = \sup\{\|\theta(v)\tilde{S}h\| : h \in H , \|h\| \leq 1\} ,$$

by definition;

$$\geq \sup\{\|\theta(v)\tilde{S}h\| : h = e_u(h_u) \in H , \|h\| = \|h_u\| \leq 1\}$$

$$\geq \sup\{\|\pi_u(\theta(v)\tilde{S}h)\| : h = e_u(h_u) , \|h_u\| \leq 1\} ,$$

by the definition of π_u ;

$$= \sup\{\|\theta_u(v)S(\pi_u h)\| : h = e_u(h_u) , \|h_u\| \leq 1\} ,$$

by (1) ;

$$= \sup\{\|\theta_u(v)Sh_u\| : h_u \in H_u , \|h_u\| \leq 1\}$$

$$= \|\theta_u(v)S\| , \text{ by definition.}$$

Hence $\|\theta_u(v)S\| \leq \|\theta(v)\tilde{S}\| \qquad (v \in V ,\ S \in B(H_u))$ . .....(2)

Finally, for v in V,

$$\|\theta_u(v)\| = \sup\{\|\theta_u(v)S\| : S \in B(H_u),\ \|S\| \leq 1\},$$

by definition;

$$\leq \sup\{\|\theta(v)\tilde{S}\| : S \in B(H_u),\ \|S\| \leq 1\},$$

by (2);

$$= \sup\{\|\theta(v)T\| : T \in B(H),\ T=\tilde{S},\ \|T\| = \|\tilde{S}\| = \|S\| \leq 1\}$$

$$\leq \sup\{\|\theta(v)T\| : T \in B(H),\ \|T\| \leq 1\}$$

$$= \|\theta(v)\| \quad \text{, by definition.}$$

Therefore, $\|\theta_u(v)\| \leq \|\theta(v)\|$ for all v in V. Now take $v = u$ to obtain, by Lemma 7.2,

$$\|u\| \leq K\|\theta_u(u)\| \leq K\|\theta(u)\| \quad \text{, or} \quad \|\theta(u)\| \geq K^{-1}\|u\| .$$

Since $u \neq 0$ was fixed, but otherwise arbitrary, it follows that $\|u\| \leq K\|\theta(u)\|$ $(u \in V)$, and so θ is injective and $\|\theta^{-1}\| \leq K$. □

If $C(\Omega)$ is separable in Theorem 7.4, which happens if and only if Ω is metrizable [19:IV.13.16], then the Hilbert space H assumes a simpler form:

7.5 COROLLARY

Let Ω be a compact metric space. There exists a separable Hilbert space H and a *-representation $\phi : C(\Omega) \longrightarrow B(H)$ such that, if $\theta : V(\Omega) \longrightarrow B(B(H))$ is defined by $\theta(x \otimes y)T = \phi(x)T\phi(y)$ for x, y in $C(\Omega)$, T in $B(H)$, then θ is a unital norm-reducing monomorphism, and $\|\theta^{-1}\| \leq K$ for an absolute constant $K < 5$.

Proof:

Since Ω is compact and metric, the space $C(\Omega)$ is separable, and hence $V(\Omega) = C(\Omega)\hat{\otimes} C(\Omega)$ is separable; for if $\{x_j\}$ is a countable dense subset of $C(\Omega)$, then $\{ \sum_1^n x_i \otimes x_j : n \in \mathbb{N} \}$ is a countable dense subset of V.

Let $W \subset V$ be countable and dense. In the proof of Theorem 7.4, define the Hilbert space H by $H = \oplus \{H_u : 0 \neq u \in W\}$; then H is a countable sum of separable Hilbert spaces, and is therefore separable. Also define $\phi = \oplus_{0 \neq u \in W} \phi_u$. For a fixed u in W, instead of a fixed u in V, the rest of the proof then carries over, provided V is replaced by W in appropriate places. At the end, it follows that $\|u\| \leq K \|\theta(u)\|$ for all u in W; by continuity this holds for all u in V, since W is dense in V. □

7.6 REMARKS

(i) The constant K (Grothendieck's constant) might be the best possible one.

(ii) If Ω is a compact metric space containing a perfect set, then the results of Section 6 show that $V(\Omega)$ is not Arens regular. For every Hilbert space H, however, the C*-algebra $B(H)$ is Arens regular (Theorem 3.7); hence, by Corollary 3.6, there can be no bicontinuous homomorphism from $V(\Omega)$ into $B(H)$, for any Hilbert space H.

(iii) In [63], Young showed that for a Banach space X, if the Banach algebra $B(X)$ is Arens regular, then X must be reflexive; he also constructed a reflexive Banach space E for which $B(E)$ is not Arens regular. For Ω compact, metric, and containing a perfect set, the Banach space X described in Proposition 7.8 below provides another example of a reflexive space for which $B(X)$ is not Arens regular; this follows from the existence of a bicontinuous monomorphism from $V(\Omega)$ into $B(X)$.

7.7 LEMMA ([56])

Let Ψ be a finite space. There exists a finite-dimensional Hilbert space H and a $*$-representation $\phi : C(\Psi) \longrightarrow B(H)$. If $\theta: C(\Psi)\hat{\otimes} C(\Psi) \longrightarrow B(B(H))$ is defined by $\theta(x\otimes y)T = \phi(x)T\phi(y)$ for x,y in $C(\Psi)$ and T in $B(H)$, then θ is a unital, norm-reducing monomorphism such that $\|u\| \leq M\,\|\theta(u)\|$ $\quad (u \in V(\Psi))$, where M is an absolute constant, $M < 10$.

Proof:

For $u \neq 0$ in $V(\Psi)$, construct the finite-dimensional Hilbert space $H_u = L^2(\Psi,\lambda)$ as in Lemma 7.2, and the $*$-representation $\phi_u : C(\Psi) \longrightarrow B(H_u)$; the unital homomorphism $\theta_u : V \longrightarrow B(B(H_u))$ satisfies $\|\theta_u\| \leq 1$, and $\|u\| \leq K\,\|\theta_u(u)\|$ as before.

By Proposition 1.17(iii), since $C(\Psi)$ is finite-dimensional, so is V. Let U be the unit sphere of V, then U is compact. In particular, there exists a finite cover of U by open balls of radius $\frac{1}{2K}$; let W be the finite set consisting of the centres of these balls.

Let $H = \oplus\{H_u : u \in W\}$ and $\phi = \bigoplus_{u \in W} \phi_u$. Since W is finite and each H_u finite-dimensional, the Hilbert space H is finite-dimensional. If θ is defined as required, then θ is a unital homomorphism and $\|\theta\| \leq 1$. As in the proof of Corollary 7.5, the proof of Theorem 7.4 carries over, and shows that $\|u\| \leq K\|\theta(u)\|$ for all u in W; thus $1 \leq K\cdot\|\theta(u)\|$ $(u \in W)$.

If now $v \in U$, then, by the definition of W, there exists u in W such that $\|u-v\| < \frac{1}{2K}$, and hence $\|\theta(u-v)\| < \frac{1}{2K}$. Then

$$\|\theta(v)\| \geq \|\theta(u)\| - \|\theta(u-v)\| \geq \frac{1}{K} - \frac{1}{2K} = \frac{1}{2K} .$$

If $0 \neq v \in V$, then $\|v\|^{-1}\cdot v \in U$, and so $\|v\| \leq 2K\cdot\|\theta(v)\|$. □

7.8 PROPOSITION

Let Ω be a compact metric space. There exists a reflexive Banach space X and a unital, norm-reducing monomorphism $\psi: V(\Omega) \longrightarrow B(X)$ such that $\|\psi^{-1}\| \leq M$ for an absolute constant $M < 10$. Thus ψ is a bicontinuous representation of $V(\Omega)$ on the reflexive space X.

Proof:

Since Ω is a compact metric space, it is separable; let $D = \{d_1, d_2, \ldots\}$ be a countable dense subset of Ω. For $n=1,2,\ldots$, let $D_n = \{d_1,\ldots,d_n\} \subset D$. Using Lemma 7.7, construct for each n a finite-dimensional Hilbert space H_n; the unital homomorphism $\theta_n : C(D_n)\hat{\otimes} C(D_n) \longrightarrow B(B(H_n))$ defined in Lemma 7.7 for each n satisfies $\|\theta_n\| \leq 1$, and $\|u\| \leq M\|\theta_n(u)\|$ $(u \in V(D_n))$.

Let X be the ℓ^2-sum of the spaces $B(H_n)$ for $n \in \mathbb{N}$; that is, X is the Banach space of sequences (T_n), $T_n \in B(H_n)$ for each n, with $\|(T_n)\|_X^2 = \sum_1^\infty \|T_n\|^2 < \infty$, and pointwise addition and scalar multiplication. Since, for each n, H_n is finite-dimensional, $B(H_n)$ is reflexive, and hence X is reflexive [34:26.8(2)].

For u in $V(\Omega)$, let u_n be the corresponding element of $V(D_n)$, for each n : as D_n is closed in Ω, $C(D_n)$ is a quotient of $C(\Omega)$; and hence $V(D_n)$ is a quotient of $V(\Omega)$ (6.11). Thus $\|u_n\| \leqslant \|u\|$, and since D is dense in Ω, $\|u\| = \sup_n \|u_n\|$.

Define $\psi : V(\Omega) \longrightarrow B(X)$ by

$$\psi(u)T = (\theta_n(u_n)T_n)_n \in X \qquad (u \in V(\Omega)\ ,\ T = (T_n) \in X).$$

Then ψ is well-defined, linear, and continuous:

$$\|\psi(u)T\|^2 = \|(\theta_n(u_n)T_n)\|^2 = \sum_1^\infty \|\theta_n(u_n)T_n\|^2 \leqslant \sum_1^\infty \|u_n\|^2 \cdot \|T_n\|^2$$
$$\leqslant \|u\|^2 \cdot \sum_1^\infty \|T_n\|^2 = \|u\|^2 \cdot \|T\|^2 \quad ,$$

and hence $\|\psi\| \leqslant 1$. It is also easy to check that ψ is a unital homomorphism, since each θ_n is one. Moreover, for u in $V(\Omega)$,

$$\|\psi(u)\|^2 = \sup \{ \sum_1^\infty \|\theta_n(u_n)T_n\|^2 : T = (T_n) \in X\ ,\ \|T\| \leqslant 1 \}$$
$$\leqslant (\sup_n \|\theta_n(u_n)\|^2) \cdot \sup \{ \|T\|^2 : T \in X\ ,\ \|T\| \leqslant 1 \}$$
$$= \sup_n \|\theta_n(u_n)\|^2 \qquad \ldots\ldots(1)$$

Also, for u in $V(\Omega)$ and each m in $\mathbb{N}$,

$$\|\psi(u)\|^2 = \sup \{ \sum_1^\infty \|\theta_n(u_n)T_n\|^2 : T = (T_n) \in X\ ,\ \|T\| \leqslant 1 \}$$
$$\geqslant \sup \{ \|\theta_m(u_m)T_m\|^2 : T = (T_n) \in X\ ,\ \|T\| \leqslant 1 \}$$
$$\geqslant \sup \{ \|\theta_m(u_m)T_m\|^2 : T = (0,\ldots,0,T_m,0\ldots) \in X\ ,\ \|T\| = \|T_m\| \leqslant 1 \}$$
$$= \sup \{ \|\theta_m(u_m)T_m\|^2 : T_m \in B(H_m)\ ,\ \|T_m\| \leqslant 1 \}$$
$$= \|\theta_m(u_m)\|^2 \quad .$$

Since this holds for each m in $\mathbb{N}$, it follows that

$\|\psi(u)\| \geq \sup_m \|\theta_m(u_m)\|$, which together with (1) implies that

$$\|\psi(u)\| = \sup_n \|\theta_n(u_n)\| \qquad (u \in V(\Omega)) .$$

Now $\|u_n\| \leq M \|\theta_n(u_n)\|$ for each u in $V(\Omega)$ and n in $\mathbb{N}$, by the definition of θ_n . Hence, for u in $V(\Omega)$,

$$\|u\| = \sup_n \|u_n\| \leq M \sup_n \|\theta_n(u_n)\| = M \|\psi(u)\| \text{ , as required .}\square$$

7.9 REMARK

The reflexive space E constructed by Young such that $B(E)$ is not Arens regular, is the ℓ^2-sum of the algebras $L^{1+\frac{1}{n}}(G)$ for G a locally compact infinite abelian group.

8. Representations on the Schatten - von Neumann classes

8.1 DEFINITION

Let H be a Hilbert space, and let $1 \leq p < \infty$. Define $C_p(H) = C_p$ by $C_p(H) = \{T \in B(H) : \sum_n \mu_n^p < \infty\}$, where (μ_n) is the decreasing sequence of non-zero eigenvalues of the positive operator $(T^*T)^{\frac{1}{2}}$, counted according to their multiplicities .

8.2 PROPERTIES

All the results mentioned can be found in Ringrose's book [46] , where the proofs are given.

Each $C_p(H) = C_p$ is a linear subspace of $B(H)$, and is a Banach space when given the norm $\|T\|_p = (\sum_n \mu_n^p)^{\frac{1}{p}}$ $(T \in C_p)$.

The spaces C_p are called the Schatten - von Neumann classes of operators. Each C_p contains the adjoint of each of its members, and is a 2-sided ideal in $B(H)$; the ideal of finite-rank operators on H is dense in each C_p .

Let C_0 be the ideal of compact operators on H , and let C_∞ denote $B(H)$. Then $C_p \subset C_0$ for $1 \leq p < \infty$. If $1 \leq p \leq q < \infty$, then $C_p \subset C_q$, and $\|\cdot\|_p \geq \|\cdot\|_q \geq \|\cdot\|_\infty$. Note also that $\|\cdot\|_0 = \|\cdot\|_\infty$.
For $1 \leq r,s,t \leq \infty$ such that $\frac{1}{r} + \frac{1}{s} = \frac{1}{t}$, and $R \in C_r$, $S \in C_s$, it is known that $RS \in C_t$, and

$$\| RS \|_t \leq \| R \|_r \| S \|_s \quad .$$

The ideal C_1 is called the ideal of trace-class operators. If T belongs to C_1 and $\{e_j\}$ is an orthonormal basis in H , let the trace of T be defined by

$$tr(T) = \sum_j < Te_j , e_j > \quad .$$

This function $tr : C_1 \longrightarrow \mathbb{C}$ does not depend on the particular orthonormal basis chosen, and has the following properties:
(i) tr is linear; (ii) $tr(S^*) = \overline{tr(S)}$ $(S \in C_1)$;
(iii) $tr(S) > 0$ if S is a positive operator in C_1 ;
(iv) $tr(AS) = tr(SA)$ $(S \in C_1 , A \in B(H))$; (v) $|tr(S)| \leq \| S \|_1 (S \in C_1)$.

The class C_2 is called the ideal of Hilbert-Schmidt operators, and is a Hilbert space with $< S , T > = tr(T^*S)$ $(S,T \in C_2)$.
If $1 < p < \infty$, then C_p is reflexive; let $1 \leq p < \infty$, and let q satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then $C_p^* \simeq C_q$ via the isomorphism

$$T \longmapsto \nu_T : C_q \longrightarrow C_p^* \quad , \text{ where } \nu_T(S) = tr(TS) \quad (S \in C_p) \; .$$

In particular, the dual of C_0 is isometrically isomorphic to C_1, and the dual of C_1 to $C_\infty = B(H)$.

Lemma 8.3 and Theorem 8.4 below are restatements of the corresponding results for $C_\infty = B(H)$ in Section 7.

8.3 LEMMA

Let Ω be a compact Hausdorff space, let $V = V(\Omega) = C(\Omega)\hat{\otimes}C(\Omega)$. For each $u \neq 0$ in V, there exists a separable Hilbert space H_u and a $*$-representation $\phi_u : C(\Omega) \longrightarrow B(H_u)$.
If $\tau_u : V \longrightarrow B(C_0(H_u))$ is defined by $\tau_u(x\otimes y)T = \phi_u(x)\cdot T\cdot\phi_u(y)$ for x,y in $C(\Omega)$ and T in $C_0(H)$, then τ_u is a unital homomorphism, $\|\tau_u\| \leq 1$, and $\|u\| \leq K\,\|\tau_u(u)\|$ for an absolute constant $K < 5$.

Proof:

Proceeding as in Lemma 7.2, let $u \neq 0$ in V, and define H_u and ϕ_u as before. Since $C_0(H_u)$ is an ideal of $B(H_u)$, it follows that $\phi_u(x)\,T\,\phi_u(y)$ belongs to $C_0(H_u)$ if T does. As with θ_u, all the other required properties of τ_u are easily checked, except for $\|u\| \leq K\,\|\tau_u(u)\|$.

Let $\varepsilon > 0$, and let $v = \sum_1^n x_j \otimes y_j$ in V be such that $\|u-v\| < \varepsilon$. Let Y be the finite-dimensional subspace of H_u spanned by the y_j's, $1 \leq j \leq n$, and let $P : H_u \longrightarrow Y$ be the projection onto Y. Since Y is finite-dimensional, P is a finite-rank operator. Recall that the procedure in the proof of Lemma 7.2 yields an element T of $B(H_u)$ such that $F(v) = \langle \theta_u(v)T(1)\,,\,1 \rangle$, where

$F \in V^*$, $F(u) = \|u\|$, and $\|T\| \leqslant K$. Then

$$F(v) = \langle \theta_u(v)T(1) , 1 \rangle = \sum_1^n \langle Ty_j , x_j^* \rangle = \sum_1^n \langle TPy_j , x_j^* \rangle$$

$$= \langle \tau_u(v)(TP)(1) , 1 \rangle ,$$

since TP is a finite-rank operator and hence compact. Also,

$$\|TP\|_{C_0} \leqslant \|T\| \cdot \|P\| \leqslant K ,$$

and hence

$$|F(v)| \leqslant \|\tau_u(v)(TP)(1)\| \leqslant \|\tau_u(v)TP\|_0 \leqslant \|\tau_u(v)\| \cdot \|TP\|$$

$$\leqslant K \|\tau_u(v)\| .$$

Again, since ε was arbitrary, it follows from the continuity of F , $\langle \cdot , \cdot \rangle$ and τ_u that $\|u\| = F(u) \leqslant K \|\tau_u(u)\|$. □

<u>8.4 THEOREM</u>

Let Ω be a compact Hausdorff space, let $V = V(\Omega) = C(\Omega) \hat{\otimes} C(\Omega)$. There exists a Hilbert space H and a $*$-representation ϕ of $C(\Omega)$ on H . If $\tau : V \longrightarrow B(C_0(H))$ is defined by

$$\tau(x \otimes y) T = \phi(x) T \phi(y) \qquad \text{for } x,y \text{ in } C(\Omega) , T \text{ in } C_0(H) ,$$

then τ is a norm-reducing unital monomorphism and $\|\tau^{-1}\| \leqslant K$ for an absolute constant $K < 5$.

<u>Proof</u>:

As in the proof of Theorem 7.4 , all that remains to be shown is that $\|u\| \leqslant K \|\tau(u)\|$ for u in V . As before, fix $u \in V$. If $S \in C_0(H_u)$, then $\tilde{S} = e_u \circ S \circ \pi_u$ is compact, and so $\tilde{S} \in C_0(H)$. Then (1) of the proof of Theorem 7.4 shows that

$$\pi_u(\tau(v)\tilde{S}h) = \tau_u(v) S (\pi_u h) \qquad (v \in V, S \in C_0(H_u) , h \in H) .$$

Hence the first chain of inequalities implies $\|\tau_u(v)S\| \leq \|\tau(v)S\|$ $(v \in V,\ S \in C_0(H))$. Similarly, the second chain of inequalities gives $\|\tau_u(v)\| \leq \|\tau(v)\|$ $(v \in V)$. Finally let $v = u$ to obtain $\|u\| \leq K\,\|\tau_u(u)\| \leq K\,\|\tau(u)\|$ by Lemma 8.3 . □

8.5 REMARKS

(i) This proof is no longer valid if $C_0(H)$ is replaced by $C_p(H)$ for some p , $1 \leq p < \infty$; it relies heavily on the equality $\|\cdot\|_0 = \|\cdot\|_{B(H)}$, while in general $\|\cdot\|_p \geq \|\cdot\|_{B(H)}$.

(ii) Corollary 7.5 remains true when $B(H)$ is replaced by $C_0(H)$.

8.6 PROPOSITION

Let Ω be a compact Hausdorff space, let $V = C(\Omega)\hat{\otimes} C(\Omega)$. Suppose that either $p = 0$ and $q = 1$, or $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

If, for some Hilbert space H and a *-representation ϕ of $C(\Omega)$ on H , there exists a norm-reducing unital monomorphism τ from V into $B(C_p(H))$, defined by $\tau(x \otimes y)T = \phi(x)\,T\,\phi(y)$ $(x, y \in C(\Omega),$ $T \in B(C_p)$) and such that $\|\tau^{-1}\| \leq K$, then there exists a unital monomorphism $\tau' : V \longrightarrow B(C_q(H))$, defined as τ for T in C_q , and satisfying $\|\tau'\| \leq 1$ and $\|(\tau')^{-1}\| \leq K$.

Proof:

Let $u = \sum_1^n x_j \otimes y_j \in V$, and suppose there exists $\tau : V \longrightarrow B(C_p)$ as described. Then $\tau(u) \in B(C_p)$; let $\tau(u)^* : C_p^* \longrightarrow C_p^*$ be the adjoint of $\tau(u)$. Let $\nu : C_q \longrightarrow C_p^*$ be the isometric isomorphism given by $\nu(T)(S) = tr(ST)$ (8.2) .

For T in C_q , let $\rho(u)T = \sum_1^n \phi(y_j)\,T\,\phi(x_j) \in C_q$.

Then, for each S in C_p, T in C_q,

$$\begin{tikzcd} C_p^* \arrow[r, "\tau(u)^*"] & C_p^* \\ C_q \arrow[u, "\nu"] \arrow[r, dashed, "\rho(u)"'] & C_q \arrow[u, "\nu"'] \end{tikzcd}$$

$$\begin{aligned} \nu(\rho(u)T)S &= \mathrm{tr}(S\cdot\sum_1^n \phi(y_j)T\phi(x_j)) \\ &= \sum_1^n \mathrm{tr}(\phi(x_j)S\phi(y_j)T) \\ &= \nu(T)(\tau(u)S) = (\tau(u)^*(\nu(T)))S \quad , \end{aligned}$$

since $\mathrm{tr}(AU) = \mathrm{tr}(UA)$ $(A \in B(H),\ U \in C_1)$: take $A = \phi(x_j)$, and $U = S\phi(y_j)T$ above.

Hence $\nu(\rho(u)T) = \tau(u)^*\ \nu(T)$ for T in C_q, and so $\nu\circ\rho(u) = \tau(u)^*\circ\nu$. Thus $\rho(u) : C_q \longrightarrow C_q$ makes the diagram above commute. Since ν is an isometry,

$$\|\tau(u)\|_{B(C_p)} = \|\tau(u)^*\|_{B(C_q)} = \|\rho(u)\|_{B(C_q)} \quad .$$

Then $K^{-1}\|u\| \leq \|\tau(u)\| = \|\rho(u)\| \leq \|u\|$, and $\rho : u \longmapsto \rho(u)$ has all the properties required, except that $\rho(x\otimes y)T = \phi(y)T\phi(x)$ for T in C_q. Let $(x\otimes y)^t = y\otimes x$ $(x\otimes y \in V)$, and define τ' by $\tau'(u) = \rho(u^t)$ for finite tensors u in V; then τ' is the map required, and $K^{-1}\|u\| \leq \|\tau'(u)\| \leq \|u\|$ for all u in V by continuity, since $\|u^t\| = \|u\|$. □

<u>8.7 COROLLARY</u>

Theorem 8.4 remains valid if $C_0(H)$ is replaced by $C_1(H)$. □

<u>8.8 REMARKS</u>

(i) In this way, in addition to the representation θ of $V(\Omega)$ on $C_\infty(H) = B(H)$ discussed in Section 7, there are also representations of the same form (and norm) on C_0 and C_1, the compact and the trace-class operators on H. As for representations on $C_p(H)$, $1<p<\infty$, the results of Section 6 show that no such conclusion is possible when $p=2$ and Ω is metric and contains a perfect set; for $V(\Omega)$ is then not Arens regular, and $C_2(H)$ is a Hilbert space, making $B(C_2(H))$ Arens regular.

If Theorem 8.4 is valid when C_0 is replaced by C_p for some $p\neq 2$, $1<p<\infty$, then Proposition 8.6 shows that it is also valid for C_q, where $\frac{1}{p}+\frac{1}{q}=1$; if so, then it might be possible to obtain the same for <u>all</u> $p\neq 2$ by using the Riesz-Thorin convexity theorem [19:VI.10.11.]. Young [63] asked whether $B(X)$ was Arens regular for X uniformly convex, or at least $X=\ell^p$, $1<p<\infty$, $p\neq 2$; if a result of the above type is true, then it would prove that $B(X)$ is not Arens regular for $X=C_p$ $(1<p\neq 2<\infty)$.

(ii) Concerning bicontinuous representations of projective tensor products of non-commutative C*-algebras on $B(H)$: despite the availability of Pisier's theorem (2.11), which replaces Grothendieck's theorem (2.1) as a tool, it seems difficult to make much progress. The problems lie precisely with the non-commutativity even in finite-dimensional cases.

9. Hermitian elements of $C(\Omega)\hat{\otimes}C(\Omega)$

The main result here, due to A.M.Sinclair, characterizes the Hermitian elements of $V(\Omega)$ for Ω a compact Hausdorff space.

9.1 DEFINITION

Let A be a unital Banach algebra. For each element a of A, define the numerical range of a in A by

$$V_A(a) = \{ f(a) : f \in A^*, \ \|f\| = f(1) = 1 \} .$$

An element a of A is said to be Hermitian if $V_A(a) \subset \mathbb{R}$; the set of Hermitian elements of A is denoted by $\mathrm{Her}(A)$.

9.2 REMARKS

(i) In general, $\mathrm{Her}(A)$ is a real closed linear subspace of A, hence a real Banach space [10:5.4]. If A is a C*-algebra, then $\mathrm{Her}(A)$ coincides with the space of self-adjoint elements of A.

(ii) Clearly, $V(\alpha a) = \alpha V(a)$ for α in $\mathbb{C}$ and a in A. It is well known [9:§10] that

$$\max \{ \operatorname{Re}\lambda : \lambda \in V(a) \} = \lim_{\alpha \downarrow 0} \frac{1}{\alpha} \{ \|1 + \alpha a\| - 1 \} \qquad (a \in A) .$$

Thus, if $a \in \mathrm{Her}(A)$, then $\max \{ \lambda : \lambda \in V(a) \} = \lim_{\alpha \downarrow 0} \frac{1}{\alpha} \{ \|1 + \alpha a\| - 1 \}$ (here $\alpha \downarrow 0$ means $\alpha \downarrow 0$ in $\mathbb{R}$).

9.3 THEOREM ([54:Remark 3.5])

Let H be a Hilbert space, let $x \in B(H)$, and let $L_x, R_x \in B(B(H))$ be the left and the right multiplication by x respectively. Then

$$\mathrm{Her}\, B(B(H)) = L_{\mathrm{Her}\, B(H)} + R_{\mathrm{Her}\, B(H)} . \ \square$$

9.4 LEMMA ("folk")

Let A and B be unital Banach algebras, and suppose $\theta : A \longrightarrow B$ is a bounded linear operator such that $\|\theta\| \leqslant 1$ and $\theta(1) = 1$. Then $\theta(\mathrm{Her}\, A) \subset \mathrm{Her}\, B$.

Proof:

Note first that for a in A, if $\mu = \max\{\mathrm{Re}\,\lambda : \lambda \in V(ia)\}$ and $\nu = \max\{\mathrm{Re}\lambda : \lambda \in V(-ia)\}$, then $\mu + \nu \geq 0$: for there exists λ_0 in $V(ia)$ such that $\mathrm{Re}\lambda_0 = \mu \Rightarrow \mathrm{Re}(-\lambda_0) = -\mu$, while $-\lambda_0 \in V(-ia)$ implies that $\mathrm{Re}(-\lambda_0) \leq \nu$; thus $-\mu \leq \nu \Rightarrow \mu + \nu \geq 0$.

Let $h \in \mathrm{Her}\,A$; the aim is to show that $\theta(h) \in \mathrm{Her}\,B$. Since h is hermitian, it follows that

$$0 = \max\{\mathrm{Re}\,\lambda : \lambda \in V(ih)\} = \lim_{\alpha \downarrow 0} \frac{1}{\alpha}\{\|1 + i\alpha h\| - 1\} \qquad .....(1)$$

Now θ is linear, and so $\theta(1 \pm \alpha h) = \theta(1) \pm i\alpha\,\theta(h) = 1 \pm i\alpha\,\theta(h)$ for $\alpha \geq 0$; from $\|\theta\| \leq 1$, for $\alpha \geq 0$,

$$\|1 \pm i\alpha\,\theta(h)\|_B - 1 = \|\theta(1 \pm i\alpha h)\|_B - 1 \leq \|1 \pm i\alpha h\|_A - 1 \;....(2)$$

Thus, for $\alpha > 0$, $\frac{1}{\alpha}\{\|1 + i\alpha\,\theta(h)\|_B - 1\} \leq \frac{1}{\alpha}\{\|1 + i\alpha h\|_A - 1\}$, and by (1), the right-hand side tends to 0 as $\alpha \downarrow 0$. If μ is defined by $\mu = \max\{\mathrm{Re}\,\lambda : \lambda \in V_B(i\theta(h))\}$, then

$$\mu = \lim_{\alpha \downarrow 0} \frac{1}{\alpha}\{\|1 + i\alpha\,\theta(h)\|_B - 1\} \leq 0 \quad .$$

Similarly, $0 = \max\{\mathrm{Re}\,\lambda : \lambda \in V(-ih)\} = \lim_{\alpha \downarrow 0} \frac{1}{\alpha}\{\|1 - i\alpha h\| - 1\}$, and hence, using (2) ,

$$\frac{1}{\alpha}\{\|1 - i\alpha\,\theta(h)\|_B - 1\} \leq \frac{1}{\alpha}\{\|1 - i\alpha h\|_A - 1\} \longrightarrow 0 \quad \text{as} \quad \alpha \downarrow 0 .$$

Hence, if $\nu = \max\{\mathrm{Re}\,\lambda : \lambda \in V_B(-i\theta(h))\}$, then $\nu \leq 0$. The remark at the beginning implies that $\mu + \nu \geq 0$, which together with $\mu \leq 0$ and $\nu \leq 0$ shows that $\mu = \nu = 0$. Therefore,

$$\max\{\mathrm{Re}\,\lambda : \lambda \in V_B(i\theta(h))\} = \max\{\mathrm{Re}\,\lambda : \lambda \in V_B(-i\theta(h)\} = 0 ,$$

and consequently $\theta(h) \in \mathrm{Her}\,B$.□

9.5 LEMMA

Let X and Y be Banach spaces, let X_0 be a proper closed subspace of X, and let Y_0 be a proper closed subspace of Y such that there exist continuous projections from X onto X_0 and from Y onto Y_0. Then $X_0 \hat{\otimes} Y + X \hat{\otimes} Y_0$ is closed in $X \hat{\otimes} Y$.

Proof:

Let $P_0 : X \longrightarrow X_0$ and $Q_0 : Y \longrightarrow Y_0$ be continuous projections. Let $P = P_0 \hat{\otimes} \mathrm{id}_Y$ and $Q = \mathrm{id}_X \hat{\otimes} Q_0$, then $P : X \hat{\otimes} Y \longrightarrow X_0 \hat{\otimes} Y$ and $Q : X \hat{\otimes} Y \longrightarrow X \hat{\otimes} Y_0$ are again continuous projections.

Now $X_0 \hat{\otimes} Y = \mathrm{Ran}\, P = \mathrm{Ker}\,(I-P)$ and $X \hat{\otimes} Y_0 = \mathrm{Ran}\, Q = \mathrm{Ker}\,(I-Q)$, where I is the identity map on $X \hat{\otimes} Y$, and $I-P$ and $I-Q$ are continuous projections. Consider $(I-P)(I-Q) : X \hat{\otimes} Y \longrightarrow X \hat{\otimes} Y$; since $PQ(x \otimes y) = P(x \otimes Q_0 y) = P_0 x \otimes Q_0 y = QP(x \otimes y)$ for $x \otimes y$ in $X \hat{\otimes} Y$, it follows that $PQ = QP$, and hence $(I-P)(I-Q) = (I-Q)(I-P)$. As shown below, this implies that $\mathrm{Ker}(I-P)(I-Q) = \mathrm{Ker}(I-P) + \mathrm{Ker}(I-Q)$, and hence $X_0 \hat{\otimes} Y + X \hat{\otimes} Y_0 = \mathrm{Ker}(I-P)(I-Q)$ is closed in $X \hat{\otimes} Y$.

The result needed to finish the proof is: if R and S are projections from a Banach space W into itself, and if $RS = SR$, then $\mathrm{Ker}\, RS = \mathrm{Ker}\, R + \mathrm{Ker}\, S$. Suppose $x \in \mathrm{Ker}\, R + \mathrm{Ker}\, S$, then $x = u+v$ and $Ru = Sv = 0$; hence $RSx = RS(u+v) = SRu = 0$, and so $x \in \mathrm{Ker}\, RS$. Conversely, if $x \in \mathrm{Ker}\, RS$, then $SRx = 0$; since R is a projection, $x = u+v \in \mathrm{Ker}\, R + \mathrm{Ran}\, R$, and hence

$0 = SRx = SRv = SR(Rw)$ for some w in W, since $v \in \mathrm{Ran}\, R$;

$= SRw = Sv$, and therefore $v \in \mathrm{Ker}\, S$. Thus $x \in \mathrm{Ker}\, R + \mathrm{Ker}\, S$. $\square$

9.6 THEOREM ([56])

Let Ω be a compact Hausdorff space. Then

$$\mathrm{Her}(C(\Omega)\hat{\otimes} C(\Omega)) = (\mathrm{Her}\ C(\Omega))\hat{\otimes}\mathbb{R}1 + \mathbb{R}1\hat{\otimes}(\mathrm{Her}\ C(\Omega)) .$$

Proof:

Let $A = C(\Omega)$. If $h \in \mathrm{Her}\,A$, then $f(h) \in \mathbb{R}$ for all f in A^* such that $\|f\| = f(1) = 1$. Suppose $F \in (A\hat{\otimes} A)^*$ and $F(1\otimes 1) = 1$, $\|F\| = 1$, then $F(h\otimes 1)$ can be interpreted as $F'(1)(h)$, where $F'(1) \in A^*$ and $\|F'(1)\| = 1 = F(1\otimes 1) = F'(1)(1)$; thus $F'(1)(h)$ belongs to $\mathbb{R}$, and so $F(h\otimes 1) \in \mathbb{R}$. Thus $h\otimes 1 \in \mathrm{Her}(A\hat{\otimes} A)$.

Similarly, $1\otimes k \in \mathrm{Her}(A\hat{\otimes} A)$ for $k \in \mathrm{Her}\,A$. Since $\mathrm{Her}(A\hat{\otimes} A)$ is a real linear space (Remark 9.2(i)) , it follows that $(\mathrm{Her}\,A)\hat{\otimes}\mathbb{R}1 + \mathbb{R}1\hat{\otimes}(\mathrm{Her}\,A) \subset \mathrm{Her}(A\hat{\otimes}A)$, and by Lemma 9.5 , this sum is closed in $\mathrm{Her}(A\hat{\otimes} A)$. (This clearly applies to any 2 unital Banach algebras A and B : $(\mathrm{Her}\,A)\hat{\otimes}\mathbb{R}1 + \mathbb{R}1\hat{\otimes}(\mathrm{Her}\,B)$ is closed in $\mathrm{Her}(A\hat{\otimes} B)$.)

For the reverse inclusion, let $u \in \mathrm{Her}(A\hat{\otimes} A)$, and let $\theta : C(\Omega)\hat{\otimes} C(\Omega) \longrightarrow B(B(H))$ be the norm-reducing unital monomorphism described in Theorem 7.4 . By Lemma 9.4 , $\theta(u) \in \mathrm{Her}\ B(B(H))$, and by Theorem 9.3 , there exist hermitian operators S and T in $B(H)$ such that $\theta(u) = L_S + R_T$.

Let $z \in H$, $\|z\| = 1$, and let $E : H \longrightarrow \mathbb{C}z$ be the (orthogonal) projection defined by $Ew = \langle w , z\rangle z$ $(w \in H)$. Then, for each U in $B(H)$, and w in H ,

$$\begin{aligned} EUE\,w &= \langle UEw , z\rangle z = \\ &= \langle U\langle w , z\rangle z , z\rangle z = \langle Uz , z\rangle \langle w , z\rangle z = \langle Uz , z\rangle Ew . \end{aligned}$$

Thus $EUE = \langle Uz , z \rangle E$ for U in $B(H)$. Hence, if $U \in B(H)$, then

$$(\theta(u)\,UE)E = L_S UE + (R_T UE)E = SUE + UETE = SUE + \langle Tz , z \rangle UE .$$

Moreover, from the definition of θ, for $x,y \in C(\Omega)$, $U \in B(H)$,

$$\theta(x \otimes y)UE = \phi(x)\,UE\,\phi(y)\,E = \phi(x)\,UE \langle \phi(y)z , z \rangle E =$$
$$= \langle \phi(y)z , z \rangle \phi(x)UE .$$

Hence, if $u = \sum_j x_j \otimes y_j$, then

$$SUE + \langle Tz , z \rangle UE = \theta(u)\,UE = \sum_j \langle \phi(y_j)z , z \rangle \phi(x_j)\,UE .$$

As $H = \{ Uz : U \in B(H) \}$, it follows that

$$S + \langle Tz , z \rangle \phi(1) = \phi\left(\sum_j \langle \phi(y_j)z , z \rangle x_j\right) ,$$

and hence $S \in \phi(C(\Omega))$. Similarly, by considering $E(\theta(u)EU)$, the operator T belongs to $\phi(C(\Omega))$. Let $S = \phi(h)$, $T = \phi(k)$ for some h and k in $C(\Omega)$. Then

$$\theta(u) = L_S + R_T = L_{\phi(h)} + R_{\phi(k)} = \theta(h \otimes 1 + 1 \otimes k) .$$

Since θ is injective, $u = h \otimes 1 + 1 \otimes k$. Define, for each s and t in Ω, the map $e_{s,t} : C(\Omega)\,\hat{\otimes}\,C(\Omega) \longrightarrow \mathbb{C}$ by

$$e_{s,t}(x \otimes y) = x(s)\,y(t) \quad (x,y \in C(\Omega)) .$$

Then $e_{s,t}$ is a linear functional on $C(\Omega)\,\hat{\otimes}\,C(\Omega)$, satisfying $\|e_{s,t}\| = 1 = e_{s,t}(1 \otimes 1)$. Since u in $C(\Omega)\,\hat{\otimes}\,C(\Omega)$ is hermitian, it follows that $e_{s,t}(u) \in \mathbb{R}$, and hence $h(s) + k(t)$ is real for all s,t in Ω. Now fix s_0 in Ω, then

$$u = (h + k(s_0)1) \otimes 1 + 1 \otimes (k + h(s_0)1) - ((h+k)(s_0))(1 \otimes 1)$$

belongs to $(\mathrm{Her}\ C(\Omega))\,\hat{\otimes}\,\mathbb{R}1 + \mathbb{R}1\,\hat{\otimes}\,(\mathrm{Her}\ C(\Omega))$: for $h(s_0) + k(s_0) \in \mathbb{R}$, and since $C(\Omega)$ is a C^*-algebra, an element x of $C(\Omega)$ is hermitian if and only if it is self-adjoint (Remark 9.2(i)), in other words if and only if x is real-valued. If $x = h + k(s_o)1$, then $x(t) = h(t) + k(s_o)$, which is real for all t in Ω; hence $h + k(s_o)1$ is in $\operatorname{Her} C(\Omega)$, and similarly $k + h(s_o)1 \in \operatorname{Her} C(\Omega)$. Thus the reverse inclusion also holds.□

9.7 COROLLARY

If Ω is a compact Hausdorff space and $V(\Omega) = C(\Omega)\hat{\otimes} C(\Omega)$, then

$$(\operatorname{Her} C(\Omega))\hat{\otimes}\mathbb{R}1 + \mathbb{R}1\hat{\otimes}(\operatorname{Her} C(\Omega)) = \operatorname{Her} V(\Omega) \subsetneq \operatorname{Her} C(\Omega)\hat{\otimes} \operatorname{Her} C(\Omega)\ .\square$$

9.8 NOTE

Using Theorem 9.6, A.M.Sinclair has proved that if A and B are unital Banach algebras, then

$$\operatorname{Her}(A\hat{\otimes} B) = \operatorname{Her} A\hat{\otimes}\mathbb{R}1 + \mathbb{R}1\hat{\otimes}\operatorname{Her} B\ .$$

REFERENCES

1. C.A.Akemann, "The dual space of an operator algebra", Trans.Amer.Math.Soc. 126 (1967), 286-302.

2. J.C.Alexander, "Compact Banach algebras", Proc.London Math. Soc.(3) 18 (1968), 1-18.

3. R.Arens, "The adjoint of a bilinear operation", Proc.Amer. Math.Soc. 2 (1951), 839-848.

4. J.W.Baker, "Compact spaces homeomorphic to a ray of ordinals", Fundamenta Mathematicae 76 (1972), 19-27.

5. S.Banach, "Théorie des Opérations Linéaires", Monografje Matematyczne, Tom 1, Warszawa, 1932.

6. M.C.F.Berglund, "Ideal C*-algebras", Duke Math.J. 40 (1973), 241-257.

7. C.Bessaga, A.Pełczyński, "Spaces of continuous functions (IV)", Studia Math. 19 (1960), 53-62.

8. F.F.Bonsall, "Compact linear operators from an algebraic standpoint", Glasgow Math.J. 8 (1967), 41-49.

9. F.F.Bonsall, J.Duncan, "Complete Normed Algebras", Springer-Verlag, Berlin, 1973.

10. F.F.Bonsall, J.Duncan, "Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras", London Math.Soc. Lecture Notes, Vol.2, Cambridge University Press, London, 1971.

11. P.Civin, B.Yood, "The second conjugate space of a Banach algebra as an algebra", Pacific J.Math. 11 (1961), 847-870.

12. J.Cusack, unpublished.

13. A.M.Davie, unpublished.

14. M.M.Day, "Normed Linear Spaces", Springer-Verlag, Berlin, 1958.

15. J.Diestel, J.J.Uhl, "Vector Measures", Amer.Math.Soc. Math. Surveys, Vol.15, Amer.Math.Soc., Providence, Rhode Island, 1977.

16. J.Dixmier, "C*-algebras", North-Holland, Amsterdam, 1977.

17. R.S.Doran, J.Wichmann, "Approximate Identities and Factorization in Banach Modules", Lecture Notes in Mathematics, Vol.768, Springer-Verlag, Berlin, 1979.

18. J.Duncan, S.A.R.Hosseiniun, "The second dual of a Banach algebra", Proc. Royal Soc. Edinburgh 84A (1979), 309-325.

19. N.Dunford, J.T.Schwartz, "Linear Operators : Part I", Interscience, New York, 1958.

20. P.Enflo, "A counterexample to the approximation problem in Banach spaces", Acta Math. 130 (1973), 309-317.

21. F.Ghahramani, "Homomorphisms and derivations on weighted convolution algebras", Ph.D. thesis, University of Edinburgh, 1978.

22. A.Grothendieck, "Critères de compacité dans les espaces fonctionnels généraux", Amer.J.Math. 74 (1952), 168-186.

23. A.Grothendieck, "Produits tensoriels topologiques et espaces nucléaires", Mem.Amer.Math.Soc., No.16, 1955.

24. A.Grothendieck, "Résumé de la théorie métrique des produits tensoriels topologiques", Boletim da Sociedade de Matemática de São Paulo 8 (1956), 1-79.

25. J.Hennefeld, "A note on the Arens products", Pacific J.Math. 26 (1968), 115-119.

26. J.G.Hocking, G.S.Young, "Topology", Addison-Wesley, Reading, Mass., 1961.

27. R.B.Holmes, "Geometric Functional Analysis and its Applications", Springer-Verlag, New York, 1975.

28. J.R.Holub, "Bounded approximate identities and tensor products", Bull.Austral.Math.Soc. 7 (1972), 443-445.

29. J.R.Holub, "Tensor product mappings", Math. Annalen 188 (1970), 1-12.

30. J.R.Holub, "Compactness in topological tensor products and operator spaces", Proc.Amer.Math.Soc. 36 (1972), 398-406.

31. B.E.Johnson, "An introduction to the theory of centralizers", Proc. London Math.Soc.(3) 14 (1964), 299-320.

32. I.Kaplansky, "Dual rings", Ann.of Math. 49 (1948), 689-701.

33. I.Kaplansky, "Normed algebras", Duke Math.J. 16 (1949), 399-418.

34. G.Köthe, "Topological Vector Spaces I", Springer-Verlag, Berlin, 1969.

35. G.Köthe, "Topological Vector Spaces II", Springer-Verlag, New York, 1979.

36. J.L.Krivine, "Sur la constante de Grothendieck", C.R.Acad. Sci. Paris, Sér.A, 284 (1977), 445-446.

37. K.Kuratowski, "Topology", Vol.I, Academic Press, New York-London, 1966.

38. J.Lindenstrauss, A.Pełczyński, "Absolutely summing operators in $\mathcal{L}_p$-spaces and their applications", Studia Math. 29 (1968), 275-326.

39. J.E.Littlewood, "On bounded bilinear forms in an infinite number of variables", Quarterly J.Math.(2) 1 (1930), 164-174.

40. R.J.Loy, "Identities in tensor products of Banach algebras", Bull.Austral.Math.Soc. 2 (1970), 253-260.

41. A.Pełczyński, Z.Semadeni, "Spaces of continuous functions (III)", Studia Math. 18 (1959), 211-222.

42. A.Pietsch, "Absolut p-summierende Abbildungen in normierten Räumen", Studia Math. 28 (1967), 333-353.

43. G.Pisier, "Grothendieck's theorem for noncommutative C*-algebras, with an appendix on Grothendieck's constant", J.Funct.Analysis 29 (1978), 397-415.

44. J.S.Pym, "The convolution of functionals on spaces of bounded functions", Proc. London Math.Soc. 15 (1965), 84-104.

45. C.E.Rickart, "General Theory of Banach Algebras", Van Nostrand, New York, 1960.

46. J.R.Ringrose, "Compact Non-Self-Adjoint Operators", Van Nostrand Reinhold Math.Stud.35, London, 1971.

47. D.A.Robbins, "Existence of a bounded approximate identity in a tensor product", Bull.Austral.Math.Soc. 6 (1972), 443-445.

48. W.Rudin, "Fourier Analysis on Groups", Wiley, New York, 1962.

49. W.Rudin, "Real and Complex Analysis", McGraw-Hill, New York, 1974.

50. W.Rudin, "Functional Analysis", McGraw-Hill, New York, 1973.

51. R.Schatten, "A Theory of Cross-Spaces", Ann.of Math.Studies 26, Princeton, 1950.

52. J.Schur, "Über lineare Transformationen in der Theorie der unendlichen Reihen", Journ.f. reine u.angew.Math. 151 (1921), 79-111.

53. I.E.Segal, "The group algebra of a locally compact group", Trans.Amer.Math.Soc. 61 (1947), 69-105.

54. A.M.Sinclair, "Jordan homomorphisms and derivations on semi-simple Banach algebras", Proc.Amer.Math.Soc. 24 (1970), 209-214.

55. A.M.Sinclair, "Cohen's factorization method using an algebra of analytic functions", Proc. London Math.Soc.(3) 39 (1979), 451-468.

56. A.M.Sinclair, unpublished.

57. E.Størmer, "Regular abelian Banach algebras of linear maps of operator algebras", J.Funct.Analysis 37 (1980), 331-373.

58. M.Takesaki, "Theory of Operator Algebras I", Springer-Verlag, New York, 1979.

59. D.C.Taylor, "The strict topology for double centralizer algebras", Trans.Amer.Math.Soc. 150 (1970), 633-643.

60. N.T.Varopoulos, "Sur les formes positives d'une algèbre de Banach", C.R.Acad.Sci., Sér. A-B, 258 (1964), 2465-2467.

61. N.T.Varopoulos, "Tensor algebras and harmonic analysis", Acta Math. 119 (1967), 51-112.

62. N.J.Young, "The irregularity of multiplication in group algebras", Quarterly J.Math. 24 (1973), 59-62.

63. N.J.Young, "Periodicity of functionals and representations of normed algebras on reflexive spaces", Proc. Edinburgh Math.Soc. 20 (1976), 99-120.